INTEGRAL EQUATION METHODS FOR NUMERICAL SOLUTION OF TWO-DIMENSIONAL LAPLACE EQUATION

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Dedicated

to the memory of our daughter

who left us even before

we could name

her

CERTIFICATE

Certified that the work contained in this thesis has been carried out under my supervision and that the work has not been submitted elsewhere for a degree.

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Bombay,
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R. S. Saxena

LIST OF FREQUENTLY OCCURING SYMBOLS

The symbols listed below are followed by a brief statement of their meaning and by the number of the page on which they are defined.

σ	Density of the single layer distribution ...	19
μ	Density of the double layer distribution ...	20
p	Point on the boundary	24
P	Any point in the domain	24
C	Closed contour	22
n	Outward normal	20
I_j	jth interval on the contour with the end points $q_{j-1/2}$ and $q_{j+1/2}$	33
q_j	Nodal point of the interval I_j	33
θ_j	Angle subtended by the interval I_j at the point p	35
θ'_j	Angle subtended by the interval I_j at the point P	38
V(P)	Value of the harmonic function V at the point P, using single layer potential ...	27
W(P)	Value of the harmonic function W at the point P using double layer potential ...	28
$\Phi(p)$	Value of V(P), when P is on the contour C ...	27
g(p)	Value of W(P), when P is on the contour C ...	28
L	Length of the contour C	30
N	Number of intervals into which the contour is divided	33
Φ^*	Torsion function	95
ψ	Conjugate torsion function	96
Ψ	Stress function	99
τ	Maximum shearing stress.	141

SYNOPSIS

This thesis deals with two methods of solving two-dimensional Laplace equation under Dirichlet conditions, numerically. The methods have their foundation in the potential theory. A given charge distribution of density σ on a closed curve C, in a plane, results in a logarithmic potential function V,

$$V(P) = -\int_C \sigma(q) \log|q-P| \, dq \qquad \ldots (i)$$

where P is a point within the domain D, bounded by the closed curve C. It satisfies two-dimensional Laplace equation in D and the boundary equation $V(P) = \Phi(p)$, where p is a point on the closed curve C. In this thesis, we shall distinguish a point on the boundary by p and a point in D by P. Thus

$$\Phi(p) = -\int_C \sigma(q) \log|q-p| \, dq \qquad \ldots (ii)$$

Conversely, if a function which satisfies Laplace equation in D and has a known value Φ on the boundary, then it is always possible to find a unique distribution $\sigma(q)$ on the boundary which satisfies (ii). Having known σ, from the given value of $\Phi(p)$, on the boundary it is possible to find the value of V(P) in D by (i) by substituting the value of $\sigma(q)$. This idea has been exploited for the numerical solution of the Dirichlet Problem for two-dimensional Laplace equation. This method has been referred to as First Method in the thesis. The first difficulty, that is encountered theoretically is the singularity

of the kernel of the integral equation (ii), but as explained in Chapter 2, the boundary integral equation for $\sigma(q)$ in (ii) is a well posed Fredholm equation of first kind. Hence the solution of the integral equation (ii) in $\sigma(q)$ is possible.

For numerical solution, one has still to manage the singularity. This has been found to be possible by approximating the arc length, adjoining the point of singularity to a straight line or to an arc of a circle as explained in Chapter 2. From the results it seems that for higher accuracy, in case of a curve with a large curvature one may approximate the arc of the curve to an arc of circle. For an accuracy of about 1 % , three or four terms of the expansion (42) (these numbers refer to the equations in the thesis) may be taken to have the desired accuracy for some suitable values of R. The formula (42) indicates that one has to be very cautious when the curvature is large. On the other hand, for a curve with a small curvature, the straight line approximation is reasonably acceptable. For more accuracy one might take only the first three terms in the second bracket of (42). Having established these results theoretically, it was considered necessary to test the results. This has been done for the case of a circle (Chapter 3) and for a rectangle (Chapter 5). The geometries of these two curves are entirely different. Further, the boundary values that were taken, was that of two harmonic functions, one of which is odd and another an even function. The analytical results are therefore known

inside the curve. These were compared with the numerical results obtained by the above method and were found to be satisfactory, although for the case of the rectangle the computation was done on a small Russian computer MINSK-2, available at I.I.T., Bombay. The results, obtained by using different quadrature formulae, in case of the circle are given in Table Nos.5-8, Chapter 3. The maximum error by Gauss Legendre quadrature formula for only 32 nodal points on the boundary turns out to be 2.4 % . Results in case of the rectangle are given in Table Nos. 34 and 36 in Chapter 5. The maximum error at any grid point is about .3 % , when 48 nodal points were taken on the boundary.

Another method is based on the analogy of the distribution of the dipoles on a curve. This leads to the normal derivative of the logarithimic kernel. For a continuous distribution of dipoles of density μ on a closed curve C, the potential function within the region enclosed by the curve C is given by the following equation

$$W(P) = \int_C \mu(q) \frac{\partial}{\partial n_q} \log |q-P| \, dq \qquad \ldots \text{(iii)}$$

which on the boundary C becomes

$$g(p) = \int_C \mu(q) \frac{\partial}{\partial n_q} \log |q-p| \, dq + \pi\mu(p) \qquad \ldots \text{(iv)}$$

where $W(P) \equiv g(p)$, at the boundary point p.

When the boundary value g(p) is given we first find the

value of $\mu(q)$ from (iv), which is a Fredholm equation of second kind having a unique solution. We then substitute back in (iii) to get the value of W(P). This method has been referred to as the Second Method in the thesis. One again encounters the types of difficulties, mentioned for the First Method. Although a simple formula can be found for the normal derivative of the kernel in the case of a circular boundary yet this result is not that easy for the case of other boundaries. However as pointed out in Chapter 2, the Cauchy-Riemann equations become very handy and pretty accurate values of the normal derivative of the kernel can be found for any boundary. The ideas were again tested in the case of the circle (Chapter 4) and rectangle (Chapter 5). The functions prescribing the boundary values were also taken to be the same. The numerical and analytical results along with the absolute errors for N = 8,16,24 and 32 in case of the circle are given in Table Nos. 21-24 in Chapter 4. The maximum error for N = 32, by Gauss Legendre quadrature formula is .14 % and by trapezoidal rule .0014 %. Similarly results were obtained in case of the rectangle which are given in Table Nos. 38 and 40. The maximum error for N = 48 is about 1 %. Thus perhaps the Second Method is slightly better than the First Method. The time consumed in running a programme on the computer IBM/7044 at I.I.T., Kanpur, in case of the circle by any of the two methods for all values of N mentioned above is less than 1 minute. The theoretical problem of errors or stability of the methods have not been discussed. There are

many other directions in which the work done in this thesis can be extended.

These methods have been used in solving some technically important problems namely the torsion problems for a beam of rectangular cross-section with a rectangular or triangular notch, in Chapter 8. This has been done after again testing the methods with reference to the problems of different geometries for which analytical results are available. This refers to the problems of beams of rectangular or equilateral triangular cross-sections (Chapter 6) and a beam with a circular cross-section with a circular notch (Chapter 7). The numerical solutions when compared with analytical ones seem to be quite satisfactory. The maximum error in case of the rectangular cross-section for $N = 48$ is .62 % by First Method and .10 % by Second Method. Similarly, in case of the equilateral triangular cross-section the maximum error for the same value of N is .65 % by First Method and .43 % by the Second Method. The errors in case of the notched circular cross-sections may be found in Chapter 7.

As pointed out earlier the torsion problems for a beam of rectangular cross-sections with notches have been done in Chapter 8. The problems have been discussed by each of the two methods separately. Firstly 16,32 and 48 nodal points were taken on the boundary. The results were obtained by the First Method. The results are converging. The same problem was again done by the Second Method, by taking the same numbers

of points. The results again converge and it is gratifying to note that they seem to converge to the same values. The work could not be carried further because of the limited capacity of the computer available. The results are given in tabular form in Chapter 8. It is thought that these are very close to the exact results.

Lastly a remark is made for the contents of Chapter 1, which describes briefly the problem and the known methods and may therefore be taken as Introduction. It appears that the methods given in this thesis are simpler than those known before and can be run on the digital computer. The programmes for all the problems are given in Appendices I to VI.

CHAPTER 1

TWO-DIMENSIONAL LAPLACE EQUATION

In most of the physical problems of elasticity, fluid mechanics, electrostatics, magnetostatics etc., one often deals with the following elliptic equation

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0 \qquad \ldots (1)$$

where Φ is a real valued twice continuously differentiable function defined on a domain X contained in R^3. The family of such class of functions is usually denoted by $C^2(X)$. When Φ is a function defined on R^2, (1) takes the following form

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0 \qquad \ldots (2)$$

A function $\Phi(x,y) \in C^2(X)$ is said to be harmonic on X iff $\Phi(x,y)$ is a solution of (2) at each point of X. In R^2, let X be a bounded point set whose interior I is simply connected and whose boundary C is a contour. If $u(x,y)$ is a prescribed function which is defined and continuous on C, then the Dirichlet Problem for the Laplace equation is that of determining a function $\Phi \equiv \Phi(x,y)$ which is defined and continuous on C, harmonic in I and identical with $u(x,y)$ on C.

It is well known that the Dirichlet Problem has a unique solution and this can be established with the help of the theories of subharmonic and super harmonic functions [34], finite

differences [15], Green's function [14], integral equations [35], Dirichlet's principle [13] and conformal mapping [45]. The solution can be analytically given in terms of the Poisson's integral if the boundary is a circle. The problem can be solved when the boundary is a rectangle with the help of the Fourier series. For any other problem the solution may be obtained if the region can be mapped conformally by an explicit mapping function onto a circular or a rectangular region. Beyond these cases the problems involved do not seem amenable for analytical treatment and recourse has to be taken to numerical methods.

We now state another type of boundary-value problem related to Laplace equation. Let X be a bounded point set in R^2, whose interior I is simply connected and whose boundary C is a smooth curve. Let $g(x,y)$ be defined and continuous on C, then the Neumann Problem for Laplace equation is to find a function $\Phi(x,y)$ which is defined and continuous on X, is harmonic on I and its normal derivative $\frac{\partial \Phi}{\partial n}$ satisfies the condition $\frac{\partial \Phi}{\partial n} = g(x,y)$ on C. It is well known that Neumann Problem will have at least one solution [35,48] only when $\int_C g \, ds = 0$. Moreover, if $\Phi(x,y)$ is such a solution then every solution is of the form $\Phi(x,y) + K$ [48], where K is an arbitrary constant. Thus under given assumptions the Neumann Problem has infinity of solutions. However, if the solution has a prescribed function value at just one point of C, then the solution exists and is unique.

Another boundary-value problem of interest is the Cauchy Problem. Let X be a bounded point set in R^2, whose interior I is simply connected and whose boundary C is a smooth curve. Let f(x,y) be a continuous function on C_1, a part of C and h(x,y) be another bounded and continuous function on the remaining part of C, say C_2, then the Cauchy Problem is to find a function $\Phi(x,y)$ which is defined and continuous on X, is harmonic on I, is identical with f(x,y) on C_1 and satisfies the condition $\frac{\partial \Phi}{\partial n} = h(x,y)$ on C_2. The fact that Cauchy Problems have unique solutions appears to have been established first by Lichtenstein [36]. Later efforts are also discussed by Miranda [42].

Because the Neumann Problem is not well posed, for a first study, we have chosen the Dirichlet Problem. In subsequent work the solution of the Neumann Problem will be attempted. Some of the numerical methods available for the purpose are the following:

1. Finite difference methods [3,17,18,21-26,38,39],
2. Variational methods [12,30,33,43,56],
3. Method of discrete Green's function [5],
4. Method of hypercircle [54],
5. Monte Carlo method [16,20,27,55],
6. Method of kernel functions [6,7,29,45],
7. Method of boundary contraction [10,11,41],
8. Method of linear programming [59].

There are other methods e.g., graphical methods [47], method of reduction to ordinary differential equations [1]; method

of approximate conformal mapping [44,57]; Newton's method [41] etc., discussed in the literature which have not been widely used. Details about these may be found in the references. A brief description of these methods is given below.

In finite difference methods, the differential equation is replaced by a difference equation and the set $C \cup I$ by a suitable discrete point set. This in essence reduces the equation in partial derivatives into one in algebra; and the region under consideration into a grid. The values of the function are found at the lattice points of the grid. If the grid is one such that the grid size h, in the x-direction is not necessarily the same as the grid size d, in the y-direction, then the Laplace equation can be reduced to [26] :

$$-20\,u_o + 2\,\frac{5-p^2}{1+p^2}\,(u_1 + u_3) + 2\,\frac{5p^2-1}{1+p^2}\,(u_2 + u_4) + (u_5 + u_6 + u_7 + u_8) = 0 \qquad \ldots (3)$$

where $p = h/d$, $u_o = u(x,y)$, $u_1 = u(x+h,y)$, $u_2 = u(x,y+d)$, $u_3 = u(x-h,y)$, $u_4 = u(x,y-d)$, $u_5 = u(x+h,y+d)$, $u_6 = u(x-h,y+d)$, $u_7 = u(x-h,y-d)$ and $u_8 = u(x+h,y-d)$.

The results can be further generalised. If (x,y) termed as the point 0 be any lattice point of the grid and the four neighbouring points $(x+h_1, y)$, $(x,y+h_2)$, $(x-h_3,y)$ and $(x,y-h_4)$ be termed as 1,2,3 and 4 respectively, then the Laplace equation at 0 can be approximated by

$$-\left[\frac{2}{h_1h_3} + \frac{2}{h_2h_4}\right]u_o + \frac{2\ u_1}{h_1(h_1+h_3)} + \frac{2\ u_2}{h_2(h_2+h_4)}$$

$$+ \frac{2\ u_3}{h_3(h_1+h_3)} + \frac{2\ u_4}{h_4(h_2+h_4)} = 0 \qquad \ldots (4)$$

where u_o, u_1, u_2, u_3, u_4 are values of u at o,1,2,3,4 respectively. This result simplifies a great deal if h_1, h_2, h_3, h_4 are all equal, and in that case,

$$u_o = \frac{(u_1 + u_2 + u_3 + u_4)}{4} \qquad \ldots (5)$$

which is discrete analogue of the mean value property for harmonic functions.

Further it may be noted that in the process of dividing the region by grid, some points of the boundary will lie on the mesh lines and not necessarily at the lattice points. If h_1, h_2, h_3, h_4 are taken to be equal then two far a point near the boundary, some or all of h_1, h_2, h_3, h_4 will be fractions of h and in that case the formula (4) will have to be applied. It may be added that except in the case of the rectangle or square it is generally not possible to have a square grid with the mesh size $h_1 = h_2 = h_3 = h_4$ and the lattice points falling on the contour C. In this case formula (5) can be usefully applied. The linear simultaneous equations are then solved by any one of the classical methods which include e.g., the Gauss method, the relaxation method and the iterative method. The method has

been very widely applied but if the value of Φ is required only at a few arbitrarily chosen fixed points within C, the whole problem is to be solved, perhaps on a finer mesh.

In variational methods for some boundary-value problems, it is possible to specify an integral expression $J(\Phi)$ for the partial differential equation. This expression can be formed for a certain class of functions Φ and which has a minimum value for just that function u, which solves the boundary-value problem. In the present case when the partial differential equation is a Laplace equation and the boundary-values are prescribed, the integral is

$$J(\Phi) \equiv \int_I\int (\Phi_x^2 + \Phi_y^2)\ dx\ dy \qquad \ldots (6)$$

which is minimised subject to the given value of Φ an C. It is proved [58] that the minimum of (6) exists. To find an approximate solution, a class of functions Φ is defined by the various sets of values of a finite number of parameters borne by a single analytical expression which assumes the required values on the boundary C for all values of the parameters. The parameter-laden expression is substituted for Φ in the integral of (6), and the minimum of $J(\Phi)$ with respect to the parameter is effected. The minimizing values of the parameters thus define that function of the given class which is required. The method is in general quite laborious in its execution and the main difficulty lies in finding a sufficiently simple function $u(x,y)$.

In the method of discrete Green's function, the exact solution of the usual difference equation that approximates the Laplace equation is directly obtained. The method applies to problems defined on a rectangle or rectangular strip, and therefore also to problems defined on regions which may be conformally mapped onto a rectangle or rectangular strip. It consists of the following procedure : (i) deriving the expression for the exact discrete Green's function satisfying the required boundary conditions (ii) evaluating this function numerically, and (iii) applying this function to obtain the desired solution of the difference equation. The Green's function is the inverse of a matrix whose elements are given by the coefficients of the difference equation and the boundary conditions, and the method could be derived and applied in matrix language without mentioning of the concept of a Green's function.

The method of hypercircle is applicable to all those boundary-value problems which can be reduced to the form where one is required to find the intersection of two orthogonal linear subspaces of a function space. The fact that the Dirichlet Problem for Laplace equation can be reduced to this form is proved by Synge [54]. To follow the method, it is necessary to be familiar with the following definitions. A hypersphere is defined as a subspace of a function space (F-space) consisting of all F-points equidistant (radius R) from some fixed F-point P (the centre). Its equation is

$$(X - P)^2 = R^2.$$

Now consider the following N equations,

$$X.\ S_\varrho = b_\varrho \qquad (\varrho = 1,2,\ldots,N)$$

where S_ϱ are N-linearly independent fixed F-vectors and b_ϱ are N fixed numbers. The F-points with position vector X satisfying these equations form a subspace which is called a hyperplane of class N and is denoted by H_N. The intersection of a hypersphere and of hyperplane of class N is known as the hypercircle of class N. Let L' and L'' be two nonintersection orthogonal linear subspaces such that $S' \in L'$ and $S'' \in L''$, then one consider the closest approach of L' and L'' by studying the square distance $(S' - S'')^2$ as S' and S'' range through L' and L'' respectively. Since L' and L'' do not intersect so $(S' - S'')^2$ never vanishes. Hence it will have a lower bound greater than zero and if we assume that this lower bound is attained for say $S' = V'$ and $S'' = V''$, then the points V' and V'' are called the vertices of L' and L'' respectively.

With these definitions we proceed to explain the method of hypercircle. Suppose, we can find a finite number of points, say (r + 1) in L' and (s + 1) in L'', then the former define a linear r-space $L'_r \subset L'$ and latter define s-space $L''_s \subset L'$. Let the vertices of L'_r and L''_s be V' and V'' respectively, then it is extremely unlikely that either of these vertices will actually be solution S of the Dirichlet Problem, but we may hope that by making r and s fairly large we may make both V' and V''

lie close to S. It is this idea which forms the basis of this method. Once we have found V' and V" then it can be shown [54] that the unknown point S must lie on a certain hypercircle having the join of V' and V" for diameter. This fact puts bounds on S^2 and in certain cases puts point wise boundes on the solution. We know then precisely the value of $(S - P)^2$, where P is the centre of the hypercircle, for its value is R^2, where R is the radius of the hypercircle. If we are interested in a good approximation, then we have to work to make R small. If this is done then P or any point on the hypercircle, is a good approximation to the solution S in the mean square sense.

We mention here some of the Monte Carlo methods of solving linear simultaneous equations without much details. The first Monte Carlo method is based on one proposed by Von Neumann and Ulam [46,55]. It is also known as direct method. There is also an adjoint method which is more suitable for finding the shape of the column vector $\vec{x}$, of unknowns than the direct method, which concentrates on a single element x_i of $\vec{x}$. Later Halton [27] studied a method of accelerating the process in adjoint method, known as sequential method.

For solving two-dimensional Laplace equation by Monte Carlo method, we replace it by finite difference approximation, which is

$$\Phi(x,y) = \frac{1}{4}\left\{\Phi(x,y+h) + \Phi(x,y-h) + \Phi(x+h,y) + \Phi(x-h,y)\right\}$$

for a mesh size h. Now suppose for simplicity that the boundary

C lies on the mesh, and consider a random walk that starts from a given interior point P of the region D, enclosed by C, and proceeds by stepping to one of the four neighbouring points at random until finally it hits the boundary C at a point Q. It may be remarked that the four possible neighbours have equal and independent probabilities at each step. Then f(Q) is an unbiased estimator of $\Phi(P)$. To show this it is enough to reduce the problem to the simultaneous linear equations. The order of H, the coefficient matrix of N x N, is equal to the number of mesh points in D and where H has four elements equal to 1/4 in each row corresponding to an interior point of D, all other elements being zero. The random walk is then identical with the first or direct method. There is an adjoint method also which means starting walks out from the boundary. If the starting point is chosen on the boundary with a probability distribution p(Q), and a walk passes through the point P just N(P) times before hitting the boundary again, then the unbiased estimator of $\Phi(P)$ is 1/4 (N(P) f(Q)/p(Q)). In view of Curtiss analysis [16] the methods turn out to be generally inefficient.

It is proved [7] that the solution of the Dirichlet Problem for two-dimensional Laplace equation in the domain D, bounded by the closed contour C can be expressed as

$$\Phi(Q) = - \int_C \Phi(P) \frac{\partial K(P,Q)}{\partial n_P} dS_P \; ; \; Q \in D, \; P \in C$$

where n_P is the outward drawn normal to the boundary at the point P and

$$K(P,Q) = N(P,Q) - G(P,Q)$$

having the following meaning :

$K(P,Q)$: kernel function,

$N(P,Q)$: Neumann's function of the domain D with respect to the given problem,

$G(P,Q)$: Green's function for the given problem.

Thus, if one can determine the kernel function K the problem is solved. Further it has been proved [7] that the kernel function can be expanded into the infinite series

$$K(P,Q) = \sum_{\mu=1}^{\infty} \Phi_{\mu}(P)\,\Phi_{\mu}(Q)$$

where $\Phi_{\mu}(P)$ is a complete orthonormal set of harmonic functions. In using this method one is particularly interested in estimates of the error committed by replacing infinite orthogonal expansions by finite ones. Nehari [45] recently has given such estimates for a number of Dirichlet Problems.

Consider a closed bounded and simply connected region R', boundary of which is a Jordan curve defined as

$$r = f(\theta') \,, \qquad (0 \leq \theta' < 2\pi) \qquad \ldots (7)$$

where $f(\theta')$ vanishes nowhere and possesses second order derivatives at all but a finite number of points. It will be further assumed that the pole can be chosen at such a point O interior to R', that the function $f(\theta')$ is single valued. Introducing a another coordinate system R,θ related to first one as follows.

$$\theta = \theta' \ ; \ R = r/f(\theta') \ ; \quad (0 \leq \theta \leq 2\pi \ ; \ 0 \leq R \leq 1) \quad \ldots (8)$$

Let $R_j = \varrho^j$; $(j = 0,1,\ldots)$ where $0 < \varrho < 1$ is a constant then the corresponding sequence of contours C_j so defined are similar figures in perspective from O. A grid system suitable for the contraction process may now be defined. A set of N equally spaced radii may be constructed emnating from O with radial angles given by :

$$\theta_n = n.\Delta\theta \ ; \quad \Delta\theta = 2\pi/N \ ; \quad (n = 0,1,\ldots\ldots,N-1)$$

where N is mostly taken as odd. The nodal points of the grid system (Fig.1,pp.18) are taken to be the points of intersection (ϱ^j,θ_n) of radial lines with the contours $\{C_j\}$ and the value of the solution $\Phi = \Phi(R,\theta)$ of Laplace equation at these points will be denoted by $\Phi(\varrho^j,\theta_n) \equiv \Phi_{j,n}$. Then in case of Dirichlet Problem, (in which boundary-values are prescribed at the boundary of R') it is implied that the numerical solution is known initially at the nodal points of C_o. Suppose now an approximating scheme can be found which relates only the unknown values on C_1 to those on C_o. The approximations relations may be either implicit or explicit, but the number of unknowns to be determined in finding the values on C_1 is in any case limited by the number of grid points on C_1 and does not involve any other unknown values on other contours. Thus to determine the values of the solution on C_1, it is required to solve at most N simultaneous equations. After the solution is determined on C_1, the original

set of values prescribed on C_o is discarded and is replaced by the newly computed data on C_1, giving rise to a new boundary-value problem on the contracted contour. The values on C_2 are now obtained from those on C_1 and the process is repeated. Now if

$$z = Rf(\theta)\ (\cos\theta + i\sin\theta)$$

then,

$$z^k = R^k \left\{f(\theta)\right\}^k (\cos k\theta + i\sin k\theta)\ ,\ k=0,1,2,\ldots$$

It is known that real and imaginery parts of z^k satisfy $\nabla^2\Phi=0$, in any bounded region of the R,θ plane. We consider them on the contour C_o i.e., $R = 1$ and write :

$$\Phi_k(1,\theta) = \left\{f(\theta)\right\}^k \cos k\theta\ ,\quad k = 0,1,\ldots$$

$$\Phi_{-k}(1,\theta) = \left\{f(\theta)\right\}^k \sin k\theta\ ,\ k = 1,2,\ldots$$

It can be proved that the functions $\left\{\Phi_{\pm k}(1,\theta)\right\}$ are linearly independent [11], and so by Gram-Schmidt process an orthogonal system $\left\{\eta_{\pm k}(1,\theta)\ ;\ k = 0,1,\ldots\right\}$ can be constructed. The functions $\eta_{\pm k}(1,\theta)$ defined only on C_o can be extended to functions $\eta_{\pm k}(R,\theta)$ on any bounded region of the R,θ plane except at $R = 0$, and it can be shown that $\nabla^2\eta_{\pm k}(R,\theta) = 0$. Now let,

$$\Phi(1,\theta) = g(\theta) \qquad \ldots (9)$$

be a continuous function defined on C_o, then a function $\Phi = \Phi(r,\theta') = \Phi(R,\theta)$ exists, harmonic in R' and tending to $g(\theta)$ on C_o as a limit point from the interior. Then it can be

proved [11] that $g(\theta)$ can be expanded as

$$g(\theta) = \sum_{-\infty}^{\infty} a_k \, \eta_k(1,\theta)$$

where

$$a_k = \int_0^{2\pi} g(\theta) \; \eta_k(1,\theta)\, d\theta$$

Consequently, the solution of the Laplace equation subject to the boundary condition (9) is then :

$$\Phi \equiv \Phi(R,\theta) = \sum_{-\infty}^{\infty} a_k \, \eta_k(R,\theta) \qquad \ldots (10)$$

where (10) is a convergent series [11] for $0 < R \leq 1$.

The method of linear programming to solve the second order partial differential equations, with prescribed boundary conditions is based upon the following principle :

An optimal solution is obtained to an over-determined system of linear inequalities that are derived from the localization of the differential equation to some set of discrete points from the prescribed conditions, and from the application of approximation formulas. We demonstrate the technique by taking a general second order partial differential equation in two independent variables s and t that holds on a closed rectangular domain R : S x T and has the form,

$$a^o\Phi + a^s\Phi_s + a^t\Phi_t + a^{ss}\Phi_{ss} + a^{st}\Phi_{st} + a^{tt}\Phi_{tt} = c \qquad \ldots (11)$$

where Φ is unknown function of s and t, Φ_s, Φ_t, Φ_{ss}, Φ_{st} and Φ_{tt} are its partial derivatives, a^o, a^s, a^t, a^{ss}, a^{tt} and c are

numerically defined continuous functions of s,t on R and a^{ss}, a^{st}, a^{tt} do not vanish on R.

Because we have five unknowns in (11), we say this is a five condition problem. Conditions may be prescribed at various point sets. For a discretization of the problem for lattice points (s_j, t_k), $j = 1(1)M$, $k = 1(1)N$, in R, we cover the possibilities rather generally by the following cases:

(i) On the lines $t = t_k$ considered, we prescribe five independent local conditions at points (s_{pk}, t_k) of the form,

$$a^o_{pk}\Phi^{pk} + a^s_{pk}\Phi^{pk}_s + a^t_{pk}\Phi^{pk}_t + a^{st}_{pk}\ \Phi^{pk}_{st} + a^{tt}_{pk}\ \Phi^{pk}_{tt} = c_{pk}\ ,$$

$$pk = 1(1)5\ ,\ k = 1(1)N \geq M \qquad \ldots (12)$$

(ii) On the lines $s = s_j$ considered, we prescribe five independent local conditions at points (s_j, t_{jp}) of the form,

$$a^o_{jp}\Phi^{jp} + a^s_{jp}\Phi^{jp}_s + a^t_{jp}\Phi^{jp}_t + a^{ss}_{jp}\ \Phi^{jp}_{ss} + a^{st}_{jp}\ \Phi^{jp}_{st} = c_{jp}\ ;$$

$$jp = 1(1)5\ ;\ j = 1(1)M \geq N \qquad \ldots (13)$$

We prescribe five conditions from (12) and (13).

The set of lattice points (s_k, t_k) considered must include all the points (s', t') at which a limited solution is required and must include pertinent prescription points (s_{jk}, t_k) and/or (s_j, t_{jp}). The difference equation (11) applied at lattice points yields

$$a^o_{jk}\Phi^{jk} + a^s_{jk}\Phi^{jk}_s + a^t_{jk}\ \Phi^{jk}_t + a^{ss}_{jk}\ \Phi^{jk}_{ss} + a^{st}_{jk}\ \Phi^{jk}_{st} + a^{tt}_{jk}\ \Phi^{jk}_{tt} = c_{jk} \qquad \ldots (14)$$

along each line $t = t_k$, $k = 1(1)\ N$, for consecutive values s_j, s_{j+1}, we have the following Taylor expansions :

$$\Phi^{jk} + h_k\Phi_s^{jk} + \frac{h_k^2\ \Phi_{ss}^{jk}}{2} - \Phi^{j+1,k} = \mathcal{E}_k^o\ ,\quad j = 1(1)N-1 \qquad \ldots (15)$$

$$h_k\Phi_s^{jk} + h_k^2\ \Phi_{ss}^{jk} - h_k\Phi_s^{j+1,k} = \mathcal{E}_{jk}^s\ ,\quad j=1(1)N-1 \qquad \ldots (16)$$

$$h_k\Phi_t^{jk} + h_k^2\ \Phi_{st}^{jk} - h_k\Phi_t^{j+1,k} = \mathcal{E}_{jk}^t\ ,\quad j=1(1)N-1 \qquad \ldots (17)$$

$$\Phi^{jk} - \Phi^{j+1,k} + h_k\Phi_s^{j+1,k} - \frac{h_k^2\ \Phi_{ss}^{k+1,k}}{2} = \mathfrak{F}_{jk}^o\ ,$$

$$j = 1(1)N-1 \qquad \ldots (18)$$

$$h_k\Phi_s^k - h_k\Phi_s^{-1,k} + h_k^2\ \Phi_{ss}^{-1,k} = \mathfrak{F}_{jk}^s\ ,\quad j=1(1)N-1 \qquad \ldots (19)$$

$$h_k\Phi_t^k - h_k\Phi_t^{-1,k} + h_k^2\ \Phi_{st}^{-1,k} = \mathfrak{F}_{jk}^t\ ,\quad j=1(1)N-1 \qquad \ldots (20)$$

where $h_k = s_{j+1} - s_j$

Similarly along the lines $s = s_j$, $j = 1(1)M$, we will get another six equations, involving E^o, E_{jk}^s, E_{jk}^t, F_{jk}^o, F_{jk}^s, and F_{jk}^t etc. We regard these two sets of equations each consisting of six equations as being homogeneous but subject to errors, which we wish to minimize. These equations along with (12) and /or (13) as pertinent together with (14), constitute an over-determined linear algebraic system whose unknowns are the values of Φ, Φ_s, Φ_t, Φ_{tt}, Φ_{st} and Φ_{ss} at the lattice points.

We define

$$\Phi_o = \max \left(\left|\mathcal{E}_k\right|, \left|\mathcal{F}_{jk}\right|, \left|E_{jk}\right|, \left|F_{jk}\right| \right)$$

and convert to a system of inequalities involving Φ_o and the solution and derivative values. Our objective is to minimize Φ_o. Computation of the linear program for which the forgoing linear model is the dual, gives by the duality principle the optimal value for Φ_o and approximate values for Φ, Φ_s, Φ_t, Φ_{ss}, Φ_{tt} and Φ_{st} at (s_j, t_k).

It would appear that in all the previous methods the techniques involved are either complicated or are not suitable for computer. In some cases for getting a partial answer the whole problem is to be solved. In the next chapter we describe integral equation method which is free from these disadvantages.

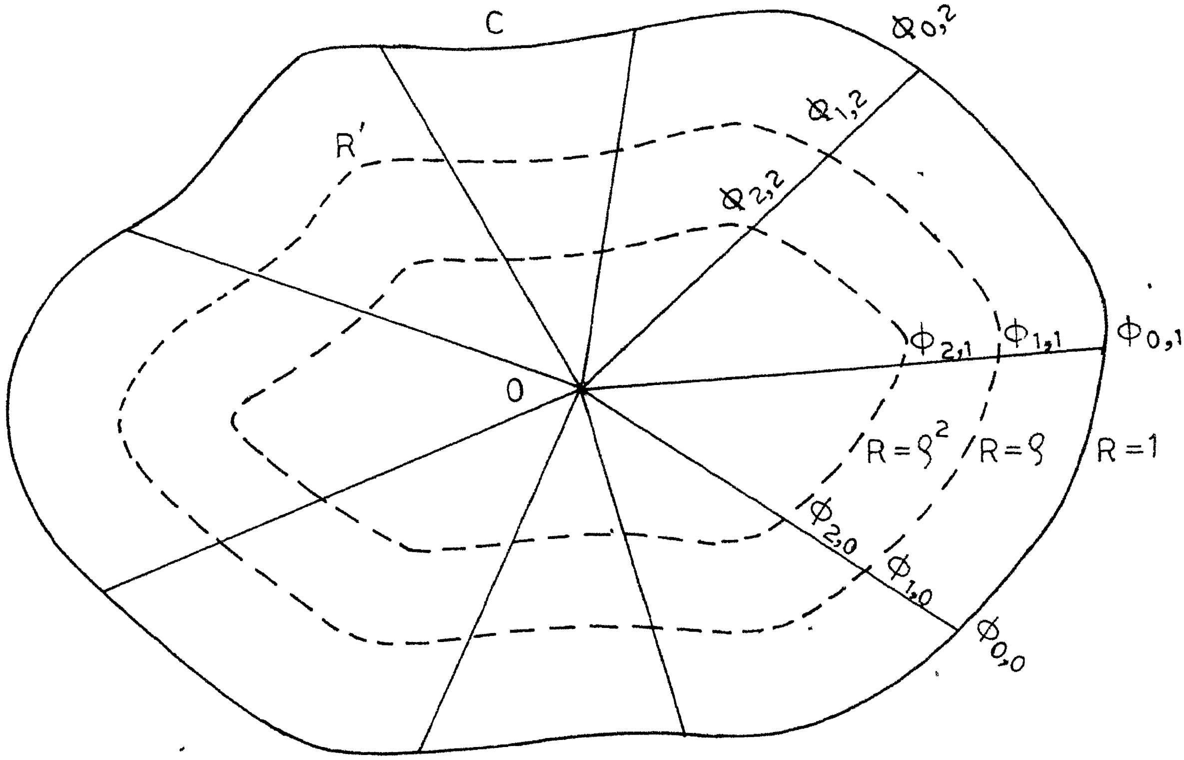

FIG. 1. GRID SYSTEM IN BOUNDARY CONTRACTION METHOD FOR LAPLACE EQUATION

CHAPTER 2

INTEGRAL EQUATION METHODS OF SOLVING TWO-DIMENSIONAL LAPLACE EQUATION

Apart from the numerical methods discussed in the last chapter, there is yet another method of solving Laplace equation, involving equations of potential theory. There are three variations of this method and are based upon Green's formula. Before entering into details of this method, we introduce some of the related definitions etc.

Potential of a Single Layer -

Let $|Q - P|$ denote the distance of a point P from a fixed point Q in the plane. The function $V(P) = -\sigma \log|Q-P|$ represents the potential at P created by placing an electric point charge σ at Q. If the charge, instead of being concentrated at a single point is continuously distributed over a domain D, with density σ, the potential at the point P(x,y) is given by

$$V(x,y) = \iint_D \sigma(\xi,\eta) \log\left(\frac{1}{|Q-P|}\right) d\xi d\eta,$$

where (ξ,η) is the point Q. Similarly, if the charge is distributed along a smooth curve C, with linear density σ, the potential is given by

$$V(x,y) = -\int_C \sigma(q) \log|q - P| \, dq \qquad \ldots (21)$$

Here q is measured along C from a fixed point on it, $\sigma(q)$ is the charge density at this point; dq the element of the arc and $|q - P|$ is the distance of P from the point q. The function V(x,y) defined as in (21) above is usually known as the single layer potential. It is known that V(x,y) satisfies the Laplace equation in two-dimension, i.e.,

$$\nabla^2 V(x,y) = 0.$$

Potential of Double Layer -

Let charges -e and +e be placed at points Q and Q' respectively. Now let Q' approaches Q along a fixed direction n, and let e increase such that

$$e.\ \overline{Q\,Q'} = M \text{ (Const.)}$$

It is well known [34] that the potential at a point P is given in the limit, by

$$W(P) \equiv W(x,y) = M.\ \frac{\partial}{\partial n_Q}\ .\ \log\left(\frac{1}{r}\right)$$

where r is the distance of P from Q and $\frac{\partial}{\partial n_Q}$ denotes differentiation with respect to Q in the direction of n . The configuration of charges just described is termed as a dipole of strength M. Now let the dipoles be distributed continuously with density $\frac{\mu}{2\pi}$ along a curve C, the direction of the dipole at each point being normal to C. In this manner we obtain a double layer which gives rise to a potential.

$$W(P) = \frac{1}{2\pi} \int_C \mu(q) \frac{\partial}{\partial n_Q} \log\left(\frac{1}{r}\right) dq_Q$$

$$= \frac{1}{2\pi} \int_0^L \mu(q) \frac{\cos(r,n)}{r} dq$$

according to the notations of Fig.2,pp. 39 . Here also q has the same meaning as in (21).

It can be easily shown that the function $W(x,y)$ is single valued and continuous with all of its derivatives at every point (x,y) not on C, and in the same domain W is harmonic i.e.,

$$\nabla^2 W(x,y) = 0$$

We shall use the symbols P_i, P_e and P_o for indicating a point inside, outside and on the contour C respectively. It is proved [37] that if P_i approaches P_o, then $W(x,y)$ approaches a finite limit $W_i(x_o,y_o)$. Similarly if P_e approaches P_o, then $W(x,y)$ approaches a definite finite limit $W_e(x_o,y_o)$. Between these quantities and $W(x_o,y_o)$ defined above the following relation holds :

$$\left.\begin{aligned} W_i(x_o,y_o) &= W(x_o,y_o) - \tfrac{1}{2}\mu(q_o) \\ W_e(x_o,y_o) &= W(x_o,y_o) + \tfrac{1}{2}\mu(q_o) \end{aligned}\right\} \quad \ldots (22)$$

or equivalently,

$$\left.\begin{aligned} W_i(x_o,y_o) + W_e(x_o,y_o) &= 2W(x_o,y_o) \\ -W_i(x_o,y_o) + W_e(x_o,y_o) &= \mu(q_o) \end{aligned}\right\} \quad \ldots (23)$$

At this stage, it is useful to remark about the behaviour of $V(x,y)$. It is well known [37] that the limit of $V(x,y)$ as $P_i \rightarrow P_o$ exists i.e.,

$$\lim_{P_i \rightarrow P_o} V(x,y) = V_i(x_o,y_o)$$

The limit of $V(x,y)$ as $P_e \rightarrow P_o$ also exists i.e.,

$$\lim_{P_e \rightarrow P_o} V(x,y) = V_e(x_o,y_o),$$

and

$$V_i(x_o,y_o) = V_e(x_o,y_o) = V(x_o,y_o)$$

i.e. $V(x,y)$ which is harmonic, remains continuous as the point $P(x,y)$ crosses the boundary C. If we compare the behaviour of $V(x,y)$ with that of $W(x,y)$, it can be seen that $V(x,y)$ behaves differently from $W(x,y)$, as the point $P(x,y)$ crosses the boundary.

Green's Function for Two-Dimensional Laplace Equation -

We consider a domain D of the xy-plane bounded by a simple closed curve C. If $P(x,y)$ and $Q(x,y)$ are every where continuous in D and piecewise continuous along C, and if D may be subdivided into a finite number of subdomains in each of which the first partial derivatives of P and Q are continuous, then

$$\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dS = \int_C (Pdx + Qdy) \qquad \ldots (24)$$

if we put [49],

$$P = - g \frac{\partial g_1}{\partial y} \quad , \quad Q = g \frac{\partial g_1}{\partial x} \quad \text{in (24)}$$

we find that

$$\int_D g \nabla^2 g_1 dS + \int_D \left(\frac{\partial g}{\partial x} \cdot \frac{\partial g_1}{\partial x} + \frac{\partial g}{\partial y} \cdot \frac{\partial g_1}{\partial y} \right) dS = \int_C g \frac{\partial g_1}{\partial n} dq \quad \ldots (25)$$

Interchanging g and g_1 in (25) and then subtracting the resulting equation from (25), we find

$$\int_D (g \nabla^2 g_1 - g_1 \nabla^2 g) dS = \int_C \left(g \frac{\partial g_1}{\partial n} - g_1 \frac{\partial g}{\partial n}\right) dq \quad \ldots (26)$$

Suppose $P(x,y)$ is a point in the interior of C. Draw a circle Γ with centre P and small radius ρ(Fig.2, pp.39) and apply (26) to the region bounded by the curves C and Γ, taking

$$g_1 = \log\left(\frac{1}{|q - P|}\right)$$

Since both g and g_1 are harmonic, it follows from (25) that

$$\left(\int_\Gamma + \int_C \right) \left\{ g(q) \frac{\partial}{\partial n} \log \frac{1}{|q - P|} - \log \frac{1}{|q - P|} \frac{\partial g}{\partial n} \right\} dq = 0 \quad \ldots (27)$$

where q is measured in the direction as shown in Fig. 2 pp. 39. It can be proved that

$$\int_\Gamma g \frac{\partial}{\partial n} \log \frac{1}{|q - P|} dq = 2\pi \; g(P) + 0(\rho)$$

and that

$$\left| \int_\Gamma \log \frac{1}{|q - P|} \frac{\partial g}{\partial n} dq \right| \leq - 2\pi K\rho \log \rho$$

where K is the upper bound of $\frac{\partial g}{\partial n}$. Putting these values into (27),

we obtain

$$g(P) = \frac{1}{2\pi} \int_C \left\{ \log \frac{1}{|q - P|} \cdot \frac{\partial}{\partial n} g(q) - g(q) \frac{\partial}{\partial n} \log \frac{1}{|q - P|} \right\} dq$$

as $\rho \longrightarrow 0$, or

$$(g,g')_P = \frac{1}{2\pi} \int_C g(q) \log' |q-P| dq - \frac{1}{2\pi} \int_C g'(q) \log |q - P| \, dq \quad \ldots (28)$$

where as stated before q denotes a current point on C and dq the the arc length of the arc at the point q, P denotes any fixed point within or without D, $|q - P|$ denotes the distance between q and P and $\log' |q - P|$ the outward normal derivative of $\log|q - P|$ at q. It can be seen from (28) that $(g,g')_P$ constitutes the potential arising from single and double layer potentials on C. Consequently, it is also a harmonic function every where except on C. It also follows that the value of g at an interior points of the region D can be determined in terms of the values of g and g' on the boundary C. Whenever the point P lies on the boundary, we shall denote it by the small letter p. Now according to Green's formula (28)

$$(g,g')_p = \frac{1}{2} g(p) \quad \ldots (29)$$

whence by virtue of the jump g/2 in the double layer potential, as we cross C inwards at p, it follows that

$$(g,g')_i = (g,g')_p + \frac{1}{2} g(p) = g(p) \quad \ldots (30)$$

where i indicates a point just inside C at p. This argument

enables us to identify $(g,g')_P$ as a representation for the harmonic function Φ, everywhere throughout D, with the exception of points on C. By virtue of the drop g/2 in the double layer potential as we cross C outwards at p, it follows that

$$(g,g')_e = (g,g')_p - \frac{1}{2}\, g(p) = 0 \qquad \ldots (31)$$

where e indicates a point just outside C at p. Further as $|P| \rightarrow \infty$

$$(g,g')_P \longrightarrow - \frac{\log |P|}{2\pi} \int_C g'(q)\,dq = 0$$

since,

$$\int_C g'(q)\,dq = 0 \qquad \ldots (32)$$

Accordingly $(g,g') = 0$ everywhere outside C. We summarize here all these results.

$(g,g')_P = \Phi(P)$, for a point everywhere in D,

$(g,g')_i = g(p)$, for a point just inside C,

$(g,g')_p = \frac{1}{2}\, g(p)$, for a point on C,

$(g,g')_e = 0$, for a point just outside C, and

$(g,g')_P = 0$, for a point everywhere outside C.

The Method of Integral Equations -

Harmonic character of g in (28) suggests a new approach to the boundary-value problems for Laplace equation. To determine a harmonic function having prescribed values g(q) on the boundary C(Dirichlet Problem) we calculate g'(q) from the

following integral equation

$$\frac{1}{2\pi} \int_C g'(q) \log|q - p| dq = \frac{1}{2\pi} \int_C g(q) \log'|q-p| dq - \frac{1}{2} g(p) \qquad \ldots (33)$$

Then on substituting g(q) and g'(q) in (28), we finally get $(g,g')_p$, the required harmonic function. The equation (33) is a Fredholm equation of first kind with the singular kernel $\log|q - p|$. It appears at first sight that this is not a Fredholm equation in the usual sense because the kernel $\log|q - p|$ involves a discontinuity at q = p. However, if L is the length of the contour, then

$$\begin{aligned} \int_C \int_C \log^2|q - p| dq\, dp &= \int_0^L \int_0^L \left\{ \log|q - p| \right\}^2 dq\, dp \\ &< \int_0^L \int_0^L (q - p)^2 dq\, dp \\ &= \frac{L^4}{6} \end{aligned}$$

which is a finite quantity and hence this equation is still a Fredholm equation [40]. Similarly, to determine the harmonic function having given g'(q) on the boundary (Neumann Problem) we calculate g(q) from (33), which is a Fredholm equation of the second kind in this case with the Kernel $\log'|q - p|$. It turns out that this equation determine g(q) only upto a constant. Afterwards $(g,g')_p$ can be obtained from (28) as in the previous case. Thus (28) combined with (33) provide a method for solving Dirichlet and Neumann boundary-value problems for Laplace equation.

It may be seen that the preceding formulations of Dirichlet and Neumann Problems involve both the single and double layer potentials in the process. But these problems can also be formulated using explicitely either a single or a double layer potential. It has already been stated earlier that a single layer potential can be expressed as

$$V(P) = - \int_C \sigma(q) \log|q - P| dq \qquad \ldots (34)$$

where V(P) is harmonic and continuous throughout the domain D, including the boundary C. Hence if we denote the value of V(P) at a boundary point p by $\Phi(p)$, then

$$\Phi(p) = - \int_C \sigma(q) \log|q - p| dq \qquad \ldots (35)$$

Thus to find the harmonic function with given boundary-values, we solve (35) with known $\Phi(p)$ and the kernel $\log|q - p|$, for $\sigma(q)$. This is then substituted in (24) to give the value of V(P) at the points P in the domain D. Note that to find V(P), we shall first have to select the point P in the domain D. For solving Neumann's Problem, the boundary equation can be obtained after differentiating (35) in the direction of the outward normal,

$$- \Phi'(p) = \frac{1}{2} \sigma(p) + \int_C \sigma(q) \log|q - p|' dq \qquad \ldots (36)$$

where $\log|q - p|'$ denotes outward normal derivative at p and hence the integral in (36) does not represent a double layer potential. Thus, for solving Neumann's Problem, where $\Phi'(p)$ is

prescribed, one has to solve (36) to get $\sigma(q)$, which on being substituted in (34), provides V(P). It turns out that the solution of (36) is not unique. This case is not elaborated further, because we shall be solving only the Dirichlet Problem.

Now we discuss a method for solving Dirichlet Problem, using double layer potential alone. As defined earlier, the double layer potential can be expressed as

$$W(P) = \frac{1}{2\pi} \int_C \mu(q) \log' |q - P| dq \qquad \ldots (37)$$

where W(P) is harmonic and continuous in the domain D except on the boundary C. Now, if we take the point P in the interior of C and denote the value of W(P) by g(p) as the point P moves towards a boundary point p, then there is a jump in the potential by the amount $\frac{1}{2}\mu(p)$, in this case. Therefore

$$g(p) = \frac{1}{2\pi} \int_C \mu(q) \log' |q - p| \, dq + \frac{1}{2} \mu(p) \qquad \ldots (38)$$

where $\log'|q - p|$ is the outward normal derivative of $\log|q - p|$ at q. Thus, if the value of g(p) is given (Dirichlet Problem), the value of $\mu(q)$ can be obtained from (38), which is a Fredholm equation of second kind. Then after substituting $\mu(q)$ obtained here, in (37), W(P) the required harmonic function can be obtained.

It is to be noted that the last two formulations involve only one integral while the first one involves two integrals. However, the first method has the advantage of giving the value of g'(q) directly once g(q) is known on the boundary and

viceversa apart from a constant for $g(q)$. This has some advantage in dealing with some physical problems e.g., in the theory of perfectly plastic solids where the interface between elastic and plastic region may be found from the continuity conditions of $g(q)$ and $g'(q)$. But from the point of view of solving Laplace equation numerically, in a given region under given boundary conditions, it appears that the second and third methods would involve less labour in comparison to first, because only one integral is to be evaluated in these methods and two integrals are to be evaluated in the first method. It may be pointed out that the first method is extensively used by Jaswon and Ponter [32] and Symm [53] for solving torsion problems in elasticity and other boundary-value problems.

In this thesis a comparative study of the last two methods is made. It is done by solving several problems having boundaries of different kinds and then comparing the computed results with the analytic ones. It may be observed that when we solve Dirichlet Problem by either of the two methods, then we have to solve integral equation of either first kind or of second kind. Specifically, we have to solve integral equation of first kind in the method using single layer potential and of second kind in the method based upon double layer potential, to be called now onwards as First Method and Second Method respectively. The integral equations in both the methods have singular kernels, but these can be taken care of.

To solve a boundary integral equation analytically is generally speaking out of question. A straight forward numerical approach is to replace the equation by a system of simultaneous linear equations referring to a set of nodal points spaced along the boundary C of the relevant domain D ; these equations are then solved for the unknown function. We give below the relevant steps.

First Method -

The first step in this method consists in replacing the integral by the sum of a finite number of terms, using any one of the quadrature formulae e.g., Simpson's rule, Trapezoidal rule or for more accuracy by Gaussian quadrature formulae. Thus, if the boundary is smooth and differentiable everywhere e.g., circle etc., then one can use Gauss-legendre quadrature formula to replace the integral into a sum of finite number of terms. In Gauss legendre's formula the limits of the integral are required to be from -1 to +1. If the limits are from o to L, as in our case, where the length of the contour C is L, we observe from (35),

$$\Phi(p) = -\int_{0}^{L} \sigma(q) \log|q - p| \, dq \qquad \ldots (39)$$

$$= -\frac{L}{2}\int_{-1}^{+1} \sigma\left(\frac{S+1}{2}L\right) \log\left|\frac{S+1}{2}L - p\right| dS \text{ ; putting } q = \frac{S+1}{2}L$$

$$= -\frac{L}{2}\sum_{j=1}^{N} w_j \, \sigma\left(\frac{S_j+1}{2} L\right) \log\left|\frac{S_j+1}{2} L - p\right| + E_G$$

where w_j, S_j and E_G are the weights, abscissae and error respectively of the Gauss-legendre quadrature formula. These values of w_j and S_j are known in the literature [52]. Putting $t_j = \frac{S_j+1}{2}L$ and neglecting error, we find

$$\Phi(p) = -\frac{L}{2} \sum_{j=1}^{N} w_j \,\sigma(t_j) \log |t_j - p|.$$

Now p is a fixed point on the boundary. We take the fixed point p as t_i, $i = 1,2,\ldots,N$ successively and obtain

$$\Phi(t_i) = -\frac{L}{2} \sum_{j=1}^{N} w_j \,\sigma(t_j) \log |t_j - t_i| \qquad \ldots (40)$$

which represents a set of N linear simultaneous equations in N unknowns. Since the right hand side of (40) contains $\log |t_j - t_i|$ which is undefined when $i = j$; an approximation for this factor is needed. It may be done in general by approximating the arc length adjoining t_i by a straight line of length ϵ, and then taking the average value of $\log |t_j - t_i|$ as follows. The average value of $\log |t_j - t_i|$ when $j = i$ is,

$$\frac{1}{\epsilon} \int_0^{\epsilon} \log S \, dS = \log \epsilon - 1 \qquad \ldots (41)$$

where ϵ is the length of one side of the interval from t_i. Thus if the interval surrounding the point P is denoted by AB (Fig.4, pp. 40) then we find the average value of $\log |t_j - t_i|$ for the interval AP and PB seperately and then take the average of these two values. In this case ϵ is the

length of AP or PB. It is natural that this approximation will be the correct value when the boundary consists of straight lines only e.g., rectangle, triangle etc., but in other cases ; this approximation may be improved by taking the arc adjoining the point t_i as an arc of a circle (Fig.5, pp.40). The radius R of this circle is the radius of curvature at t_i . Let the arc subtend an angle α at the centre of curvature, then the average value of $\log |t_j - t_i|$ will be I_ϵ / ϵ, where ϵ is the length of the approximating arc and

$$I_\epsilon = \int_{t_i}^{t} \log |t_j - t_i| \, dt_j ,$$

$$= R \int_0^\alpha \log (2R \sin \frac{\theta}{2}) \, d\theta$$

$$= R \int_0^\alpha \log \; 2R \left(\frac{\theta}{2} - \frac{\theta^3}{48} + \frac{\theta^5}{3840} - \dots \right) d\theta .$$

$$= \epsilon(\log \epsilon - 1) - \left(\frac{\epsilon^3}{72R^2} + \frac{\epsilon^5}{14400R^4} + \frac{\epsilon^7}{1270080 \; R^6} + \dots\right) \qquad \dots (42)$$

It may be seen here that if the curvature at the point t_i is very large, then the terms in the second bracket shall make substantial contribution. Otherwise correction may be obtained by one or two terms in the second bracket of (42).

It can be proved that the coefficient matrix in (40) is non singular and consequently its solution exists i.e., we can find $\sigma(t_j)$; $j=1,2,\dots,N$. Then as a second step, we approxi-

mate the integral in (34), using the same quadrature formula, thus

$$V(P) = - \int_0^L \sigma(q) \log |q - P| \, dq$$

$$= - \frac{L}{2} \int_{-1}^{+1} \sigma\left(\frac{S+1}{2} L \right) \log \left| \frac{S+1}{2} L - P \right| dS$$

$$= - \frac{L}{2} \sum_{j=1}^{N} w_j \, \sigma(t_j) \log |t_j - P| \qquad \dots (43)$$

It may be observed that the logarithmic factor in this equation does not need any approximation, since the point P is inside the contour. Substituting the value of $\sigma(t_j)$ obtained from (40) and the values of w_j and t_j from tables [52] into (43), we can compute the required value of V(P) by a quadrature formula.

At this stage it is useful to remark that the process in (39) can be further simplified by dividing the contour length into N equal intervals and taking the middle point of each interval as its nodal point. It is assumed that σ is constant on each of the interval. Consequently, equation (39) takes the form

$$\Phi(p) = - \sum_{j=1}^{N} \sigma(q_j) \int_{q_j - \frac{1}{2}}^{q_j + \frac{1}{2}} \log |q - p| \, dq \qquad \dots (44)$$

where q_j is the nodal point of the interval $I_j \equiv (q_j - \frac{1}{2}, q_j + \frac{1}{2})$, fig.3, pp.39. Now if h is the length of each interval, then

the integrals on the right hand side of the above equation can be easily evaluated. Replacing p by q_i, $i = 1,2,..,N$, we get a set of N linear simultaneous equations for N unknowns $\sigma(q_j)$. The values of $\sigma(q_j)$ are substituted in (39) after expressing it into the form (44) with p replaced by P on its right hand side. This sum on the right hand side is the harmonic function V(P).

Second Method -

In this method, the Fredholm equation of second kind is involved. But the techniques used in the First Method can be extended. The kernel in this case is different from that in the first case where it is $\log|q - p|$. We again divide the contour into N equal parts. We suppose that $\mu(q)$ is constant in each interval. Regarding $\log'|q - p|$, we might suppose that it is also constant in each interval, in which case it is evaluated as follows.

$$\log'|q - p| = \frac{\partial}{\partial n_q} \log|q - p|$$

$$= \frac{(x-x_1)\cos\beta + (y-y_1)\sin\beta}{(x-x_1)^2 + (y-y_1)^2} \qquad \ldots (45)$$

where (x,y) and (x_1,y_1) are coordinates of the points q and p respectively and β is the angle of inclination of the outward normal to q to the x-axis (Fig.7,pp.50). If however $\log'|q-p|$ is supposed to be variable, we can get a rather more accurate value. In this case the well known Cauchy Riemann equations

are used which state that $\frac{\partial}{\partial n_q} \log|q - p| = \frac{\partial \theta}{\partial q}$, where θ is the angle which the radius vector $|q - p|$ makes with any line fixed in the plane, whence for the interval $(q_{j-1/2}, q_{j+1/2})$, $\int \frac{\partial}{\partial n_q} \log|q - p| \, dq$ may be computed as the change in θ as the point q moves from one end of the interval to the other.

After dividing the contour length into N intervals and assuming μ to be constant in each of these intervals we have from equation (38)

$$g(p) = \frac{1}{2\pi} \sum_{j=1}^{N} \mu(q_j) \int_{q_{j-1/2}}^{q_{j+1/2}} \log'|q-p| \, dq + \frac{1}{2} \mu(p)$$

$$= \frac{1}{2\pi} \sum_{j=1}^{N} \mu(q_j) \left(\theta_{j+1/2} - \theta_{j-1/2}\right) + \frac{1}{2} \mu(p) \qquad \ldots (46)$$

where $\theta_{j-1/2}$ and $\theta_{j+1/2}$ are the inclinations of the radii vectors joining p and $q_{j-1/2}$ and p and $q_{j+1/2}$ respectively with respect to any fixed line in the plane. Here as explained earlier q_j is the middle point of the interval I_j with end points $q_{j-1/2}$ and $q_{j+1/2}$. Thus if we write $\theta_j = \theta_{j+1/2} - \theta_{j-1/2}$ and $\mu_j = \mu(q_j)$,

$$g(p) = \frac{1}{2\pi} \sum_{j=1}^{N} \mu_j \theta_j + \frac{1}{2} \mu(p).$$

Replacing p by q_i, $i = 1,2,\ldots,N$ in the above equation

$$g(q_i) = \frac{1}{2\pi} \sum_{j=1}^{N} \mu_j \theta_j + \frac{1}{2} \mu_i$$

or

$$2\pi \, g(q_i) = \sum_{j \neq i} \mu_j \theta_j + (\pi + \theta_i) \mu_i \qquad \ldots (47)$$

Since in the Dirichlet Problem $g(q_i)$ is already given (47) represents a system of N linear simultaneous equations with the following coefficient matrix.

$$D_N = \begin{bmatrix} (\pi + \theta_1) & \theta_2 & \theta_3 & \cdots\cdots & \theta_N \\ \theta_1 & (\pi + \theta_2) & \theta_3 & \cdots\cdots & \theta_N \\ \theta_1 & \theta_2 & (\pi + \theta_3) & \cdots\cdots & \theta_N \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \theta_1 & \theta_2 & \theta_3 & \cdots\cdots & (\pi + \theta_N) \end{bmatrix}$$

It is obvious from the definition of θ_j that

$$\sum_{j=1}^{N} \theta_j = \pi \qquad \ldots (48)$$

Now we shall prove that the rank of the matrix D_N is N. Consider, the linear sum

$$C_1 X_1 + C_2 X_2 + C_3 X_3 + \ldots\ldots + C_N X_N = 0$$

where each X_i denotes a row vector of the matrix D_N i.e.,

$$X_i = (\theta_1, \theta_2, \ldots\ldots \theta_{i-1}, (\pi + \theta_i), \theta_{i+1}, \ldots \theta_N)$$
$$i = 1,2,\ldots\ldots,N$$

and C_1, $C_2, \ldots . C_N$ are arbitrary constants. Thus

$$\Big\{ C_1(\pi + \theta_1) + C_2\theta_1 + \ldots C_N\theta_1,\ C_1\theta_2 + C_2(\pi+\theta_2) + \ldots C_N\theta_2, \ldots$$
$$C_1\theta_N + C_2\theta_N + \ldots . + C_N(\pi + \theta_N) \Big\} = 0$$

which implies that

$$C_1\pi + (C_1 + C_2 + \ldots\ldots + C_N)\theta_1 = 0$$

$$C_2\pi + (C_1 + C_2 + \ldots\ldots + C_N)\theta_2 = 0$$

$$\vdots \qquad \ldots (49)$$

$$C_N\pi + (C_1 + C_2 + \ldots\ldots + C_N)\theta_N = 0$$

Adding all of them, we find

$$\pi(C_1 + C_2 + \ldots + C_N) + (C_1 + C_2 + \ldots\ldots + C_N)\sum_{j=1}^{N} \theta_j = 0$$

or $$2\pi \sum_{j=1}^{N} C_j = 0 , \quad \text{from (48)}$$

or $$\sum_{j=1}^{N} C_j = 0$$

Hence it follows from (49) that $C_1 = C_2 = \ldots = C_N = 0$ consequently, the matrix D_N is non-singular and the solution of (47) exists. Having obtained, the values of $\mu(q_j)$ we proceed to find W(P) at any point P inside the domain D. From (37), we see that

$$W(P) = \frac{1}{2\pi} \sum_{j=1}^{N} \mu(q_j) \int_{q_{j-1/2}}^{q_{j+1/2}} \log' |q - P| \, dq$$

$$= \frac{1}{2\pi} \sum_{j=1}^{N} \mu(q_j)(\theta'_{j+1/2} - \theta'_{j-1/2}) \qquad \ldots (50)$$

where $\theta'_{j-1/2}$ and $\theta'_{j+1/2}$ are the inclinations of the radii vectors joining P and $q_{j-1/2}$ and P and $q_{j+1/2}$ with respect to

any fixed line in the plane. Putting $\theta_j' = \theta'_{j+1/2} - \theta'_{j-1/2}$ and $\mu(q_j) = \mu_j$,

$$W(P) = \frac{1}{2\pi} \sum_{j=1}^{N} \mu_j \, \theta_j' \qquad \ldots (51)$$

substituting the values of μ_j, obtained from (47) in the above equation, we obtain finally W(P). It may be seen that since the point P is inside the contour so

$$\sum_{j=1}^{N} \theta_j' = 2\pi \qquad \ldots (52)$$

Equations (48) and (52) may be used as checks for the values of θ_j and θ_j' respectively.

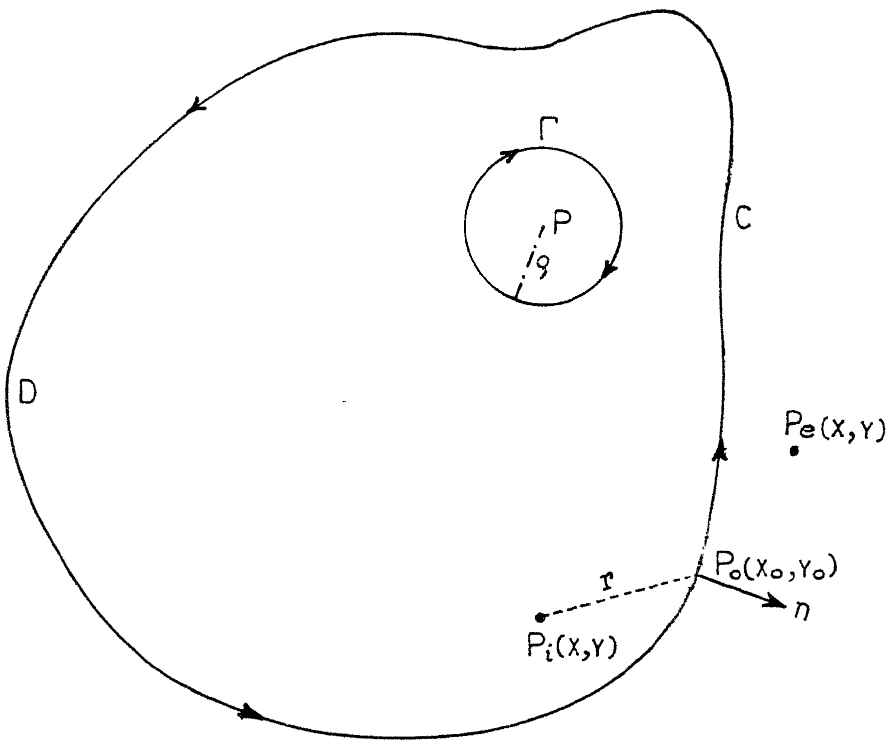

FIG. 2 - P_i, P_e AND P_o DENOTE THE POINT P INSIDE, OUTSIDE AND ON THE CONTOUR C.

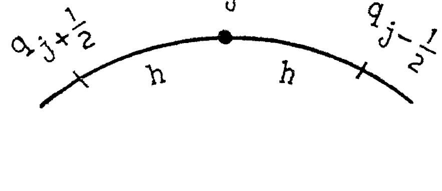

FIG. 3 - DOT INDICATING THE NODAL POINT AND DASHES THE END POINTS OF THE INTERVAL $I_j \equiv (q_j - \frac{1}{2}, q_j + \frac{1}{2}$

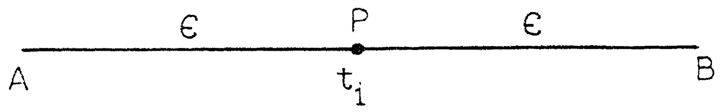

FIG. 4 - APPROXIMATION OF THE ARCS ADJOINING THE NODAL POINT t_i BY STRAIGHT LINES OF LENGTH ϵ EACH.

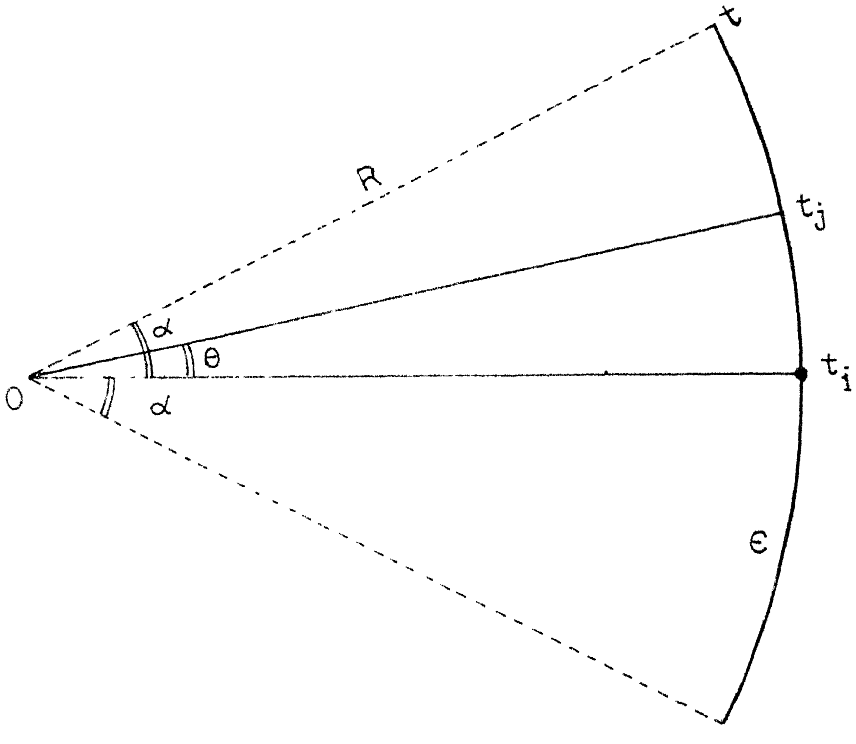

FIG. 5 - APPROXIMATION OF THE ARCS ADJOINING THE NODAL POINT t_i BY THE ARCS OF THE CIRCLE OF RADIUS R OF EQUAL LENGTH.

CHAPTER 3

DIRICHLET PROBLEM FOR A CIRCULAR DISC BY FIRST METHOD

As a first step the ideas have been checked with reference to a circular boundary. This has the advantage of constant curvature. As mentioned earlier, for a given $\Phi(p)$ we have first to solve

$$\Phi(p) = - \int_C \sigma(q) \log|q - p| dq , \qquad \ldots (53)$$

for $\sigma(q)$. For this the integral is replaced by the sum of a finite number of terms. This was done by approximating the integral by three well known quadrature formulae, namely Gauss-Legendre quadrature formula, Lobatto quadrature formula and trapezoidal rule. We give some details of each of these methods very briefly.

Let the integral to be evaluated be $\int_a^b w(x) f(x) dx$. We assume that $f(x) \in C^{2N}[a,b]$ and $w(x) \geq 0$. The latter is called the weight function and is defined on $[a,b]$. It is well known that corresponding to $w(x)$ and interval $[a,b]$, a set of orthogonal polynomials $\{p_N(x)\}$ can be defined [28]. If the zeros of $p_N(x)$ be x_i, $i=1,2,\ldots,N$, then $a < x_1 < x_2 \ldots\ldots < x_N < b$ and that the positive constants w_i, $i=1,2,\ldots,N$ can be found such that [28]

$$\int_a^b w(x) f(x) dx = \sum_{k=1}^{N} w_k f(x_k) + \frac{f^{2N}(\lambda)}{(2N)!\, K_N^2} \; ; \quad a < \lambda < b \qquad \ldots (54)$$

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$$\int_a^b w(x) f(x) dx = \sum_{k=1}^{N} w_k f(x_k) + \frac{f^{2N}(\lambda)}{(2N)!\, K_N^2} \; ; \quad a < \lambda < b \qquad \ldots (54)$$

It can be easily proved [28] that

$$\sum_{k=1}^{N} w_k = b - a \qquad \ldots (55)$$

In case $w(x) \equiv 1$, $a = -1$, $b = +1$; the formula obtained is known as Gauss-Legendre quadrature formula. It is an open type quadrature formula and has degree of precision $(2r-1)$. It may be mentioned that if a quadrature formula yields exact results when $f(x)$ is an arbitrary polynomial of degree r or less, but fails to give exact result for at least one polynomial of degree $(r+1)$, it is said to possess a degree of precision equal to r.

In Lobatto quadrature formula, we evaluate the integral $\int_{-1}^{1} f(x)\,dx$. Assume that $f(x) \in C^{2N-2}[-1,+1]$. The Lobatto quadrature formula states :

$$\int_{-1}^{1} f(x)\,dx = \frac{2}{N(N-1)} \left\{ f(1)+f(-1) \right\} + \sum_{j=2}^{N-1} w_j f(x_j) + E_L \qquad \ldots (56)$$

where x_j is the $(j-1)$th zero of $P'_{N-1}(x)$, where $P_N(x)$ is the Legendre polynomial of order N and

$$E_L \equiv E_L(f) = \frac{-N(N-1)^3 \cdot 2^{2N-1} \cdot \{(N-2)!\}^4}{(2N-1) \cdot \{(2N-2)!\}^3} f^{2N-2}(\lambda), \; -1 < \lambda < 1$$

Therefore the degree of precision is $(2r-3)$. This formula is of the closed type and is sometimes helpful especially when $f(x)$ displays a peculiar behaviour at $x = \pm 1$, such as an apparent singularity etc. or when $f(\pm 1) = 0$. In the latter

case, the degree of precision is $(2r+1)$, whereas in Gauss formula, the use of r ordinates will still lead to a degree of precision $(2r-1)$.

As a last integration formula, we have taken the well known trapezoidal rule. Under certain conditions, the trapezoidal rule gives surprisingly good results [19] when it is applied to periodic functions- much better in fact than what might have been predicted from error estimate. Denoting by T_N the N-point trapezoidal rule, which states :

$$\int_a^b f(x)\,dx = \frac{h}{2}\left\{f(a)+2f(a+h)+\ldots\ldots 2f(a+(N-1)h)+f(b)\right\} + E_{T_N}$$

where $h = \frac{b-a}{N}$ and E_{T_N} is the error in the N-point trapezoidal rule. In case $f(x) \in C^{2k+1}[a,b]$ and $f(x)$ is a periodic function, then

$$\left| E_{T_N} \right| = \left| \int_a^b f(x)\,dx - T_N(f) \right| \leq \frac{K}{N^{2k+1}} ,$$

where K is a constant, independent of N. Under these conditions noting that for periodicity $f(a) = f(b)$, the trapezoidal rule takes the form

$$T_N = h \sum_{k=1}^{N} f(a + (k-1)h)$$

consequently, if $p = b-a$ then

$$\int_a^b f(x)\,dx = \int_o^p f(x)\,dx = \frac{p}{N} \sum_{k=1}^{N} f\left(\frac{k-1}{N}\, p\right) \qquad \ldots (57)$$

There are many other quadrature formulae available in the litera-

ture e.g., Simpson's rule, the formulae based upon finite differences etc. We have chosen Gauss-Legendre and Lobatto quadrature formulae because of their higher accuracy and trapezoidal rule because of its simplicity. The latter has given exceptionally good results at least for the cases that we have discussed.

In this chapter we have taken the case of a circular disc of radius $\frac{1}{\pi}$, just to avoid the repetition of the factor π. Two cases are considered. In one case the value of the function on the boundary is given by $\Phi(p) = x$ and in the other by $\Phi(p) = x^2 - y^2$. These functions have been chosen because of their simplicity and also because one is an odd function and another is even. The centre of the disc is the origin of the coordinate system, thus the equation of its boundary which is a circle is $x^2 + y^2 = 1/\pi^2$.

For using Gauss-Legendre quadrature formula in (53), we put

$$q = s + 1, \quad dq = ds \qquad \ldots (58)$$

thus

$$\Phi(p) = -\int_{-1}^{1} \sigma(s+1) \log |s+1 - p| \, ds \qquad \ldots (59)$$

Replacing p by p_i and substituting for $\Phi(p) = x$, as a first case in the above equation, we get

$$x_i = -\int_{-1}^{1} \sigma(s+1) \log | s+1-p_i| \, ds \qquad \ldots (60)$$

where p_i is the point (x_i, y_i) on the boundary of the disc. Replacing the integral in (60), using (54), we get

$$x_i = - \sum_{j=1}^{N} w_j \, \sigma(s_j+1) \log |s_j+1- p_i| \qquad \ldots (61)$$

where w_j and s_j are the weights and abscissas respectively of the Gauss-formula, available in tables [52]. The nodal points on the boundary were taken as the fixed points i.e.,

$$p_i = s_i+1 \ , \ i = 1,2,\ldots\ldots, N$$

As stated in the previous chapter the coefficient matrix of the system of linear simultaneous equations in (61) is non-singular. Crout's method was used to solve it. Values of σ's for N = 8,16,24 and 32 are shown in Table Nos. 1-4,pps. 51-53 and compared with its analytic values, which are obtained as follows.

Let the potential function on opposite sides of a curve C(fig.6,pp. 50) be denoted by Φ_1, Φ_2 ; the normal by n and if σ is continuous on C, which itself has continuous curvature, then on the curve [49]

$$\Phi_1 = \Phi_2$$

and

$$\left\{ \frac{\partial \Phi_1}{\partial n} - \frac{\partial \Phi_2}{\partial n} \right\}_p = 2\pi \, \sigma(p) \qquad \ldots (62)$$

For the harmonic function $\Phi(p) = x = r \cos \theta$, in case of the circle of radius a, $\Phi_1 = r \cos \theta$, $\Phi_2 = \frac{a^2 \cos \theta}{r}$ define the

potential functions on opposite sides of the arc of the circle. Since the normal coincides with the radius vector, therefore from (62), for any point (a,θ) on the boundary

$$2\pi\ \sigma(a,\theta) = \left[\frac{\partial}{\partial r}\ (r \cos\theta) - \frac{\partial}{\partial r}\ \left(\frac{a^2 \cos\theta}{r} \right) \right]_{(a,\theta)} = 2\cos\theta$$

In the present case, where $a = 1/\pi$

$$\sigma(q) = \frac{\cos\theta}{\pi} \qquad \ldots (63)$$

Similarly, when $\Phi(p) = x^2 - y^2$, it can be proved that in case of the circle of radius $1/\pi$

$$\sigma(q) = \frac{2}{\pi^2} \cos 2\theta \qquad \ldots (64)$$

Having found $\sigma(q)$, we proceed to find the value of $V(P)$ inside the region. We replace p by P and $\Phi(p)$ by $V(P)$ in (59). Thus

$$V(P) = -\int_{-1}^{1} \sigma(s+1)\ \log|s+1-P|\ ds$$

$$= -\sum_{j=1}^{N} w_j\ \sigma(s_j+1) \log|s_j+1-P|$$

$$= -\sum_{j=1}^{N} w_j\ \sigma(s_j+1) \log\left\{ (x_j-\xi)^2+(y_j-\eta)^2 \right\}^{1/2}$$

$$\ldots (65)$$

where P is the point (ξ,η) and (x_j,y_j) are the coordinates of the point on the curve corresponding to the arc length s_j+1. Substituting in (65) the values of σ's, obtained from (61) and

w_j from tables [52], we finally get V(P).

The value of V(P) was evaluated at eight points (fig.7, pp. 50), one in each quadrant and two points one on each axis, one on the positive side and other on the negative, inside the circle. These values appear in Table Nos. 5-8, pps. 54,55 along with the corresponding analytic values for values of N mentioned earlier. With this method the maximum error at these points for N = 32 in this case i.e., when $\Phi(p) = x$ is 1.63 % .

For Lobatto quadrature formula only weights and abscissas, as given in tables [52] are to be changed in equations (61) and (65). Computed and analytic values of σ's in this case appear in Table Nos.1-4, pps. 51-53 and those of V(P) in Table Nos.5-8,pps. 54, 55 . The maximum error in V(P) for N = 32, at any grid point is 1.68 % .

Finally the trapezoidal rule can be applied more easily as follows. Applying (57) in (53) after writing it as

$$\Phi(p) = -\int_0^2 \sigma(q)\log|q-p|\,dq\ , \qquad \ldots (66)$$

we obtain

$$\Phi(p_i) = -\frac{2}{N}\sum_{k=1}^{N}\sigma\left(\frac{2(k-1)}{N}\right)\log\left|\frac{2(k-1)}{N} - p_i\right| \ \ldots (67)$$

where p is replaced by p_i in (66). We take a fixed point on the contour as the starting point from where the length is measured and then take

$$p_i = \frac{2(i-1)}{N}, \quad i = 1,2,\ldots,N.$$

Since we are considering the case where $\Phi(p) = x$, so

$$x_i = -\frac{2}{N}\sum_{k=1}^{N} \sigma_k \log\left|\frac{2(k-1)}{N} - \frac{2(i-1)}{N}\right| \qquad \ldots (68)$$

where (x_i, y_i) are the coordinates of the point on the contour corresponding to the arc length $2(i-1)/N$ and $\sigma_k = \sigma\left(\frac{2(k-1)}{N}\right)$. The coefficient matrix of the system of linear equations in (68) can be proved to be non-singular one. Crout's method was used to solve this system and the values of σ's along with its analytical values are given in Table Nos. 1-4, pps. 51-53.

Lastly

$$V(P) = -\int_0^2 \sigma(q)\log|q-p|\,dq$$

$$= -\frac{2}{N}\sum_{k=1}^{N} \sigma\left(\frac{2(k-1)}{N}\right)\log\left|\frac{2(k-1)}{N} - p\right|$$

$$= -\frac{1}{N}\sum_{k=1}^{N} \sigma_k \,.\, \log\left\{(x_k-\xi)^2+(y_k-\eta)^2\right\} \qquad \ldots (69)$$

where (ξ, η) are the coordinates of the point P inside the circle. Substituting the values of σ's, obtained from (68), V(P) was computed at the same eight points for N= 8,16,24 and 32. These values along with the error appear in Table Nos.5-8, pps. 54, 55 and the maximum error for N = 32 at any of the points mentioned earlier is 2.40 % .

Similar calculations were done for the other case when

$\Phi(p) = x^2 - y^2$. Values of σ's obtained using the three different quadrature formulae are given in Table Nos. 9-12, pps. 56-58 . Analytic values of σ's in this case were calculated from (64). Computed as well as analytic values of V(P) at all those eight points for values of N, mentioned earlier are shown in Table Nos. 13-16, pps. 59, 60 . The maximum error for N = 32 at any of these points by Gauss-Legendre quadrature formula is 2.44 % , by Lobatto quadrature formula 2.65 % and by trapezoidal rule 6.47 % . Looking into the percentages of error in both problems, using different quadrature formulae it is seen that Gauss-Legendre formula is more effective. Entire computational work was done on the computer IBM/7044, at I.I.T., Kanpur and a single program, each for Gauss-Legendre quadrature formula and trapezoidal rule when $\Phi(p) = x$ is given in Appendix I.

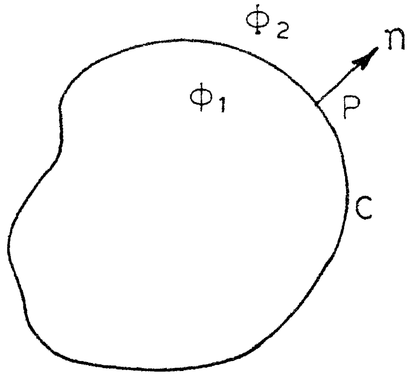

FIG. 6 - ϕ_1 AND ϕ_2 DENOTE THE POTENTIAL FUNCTIONS INSIDE AND OUTSIDE THE CURVE C RESPECTIVELY AND n THE DIRECTION OF THE NORMAL AT P.

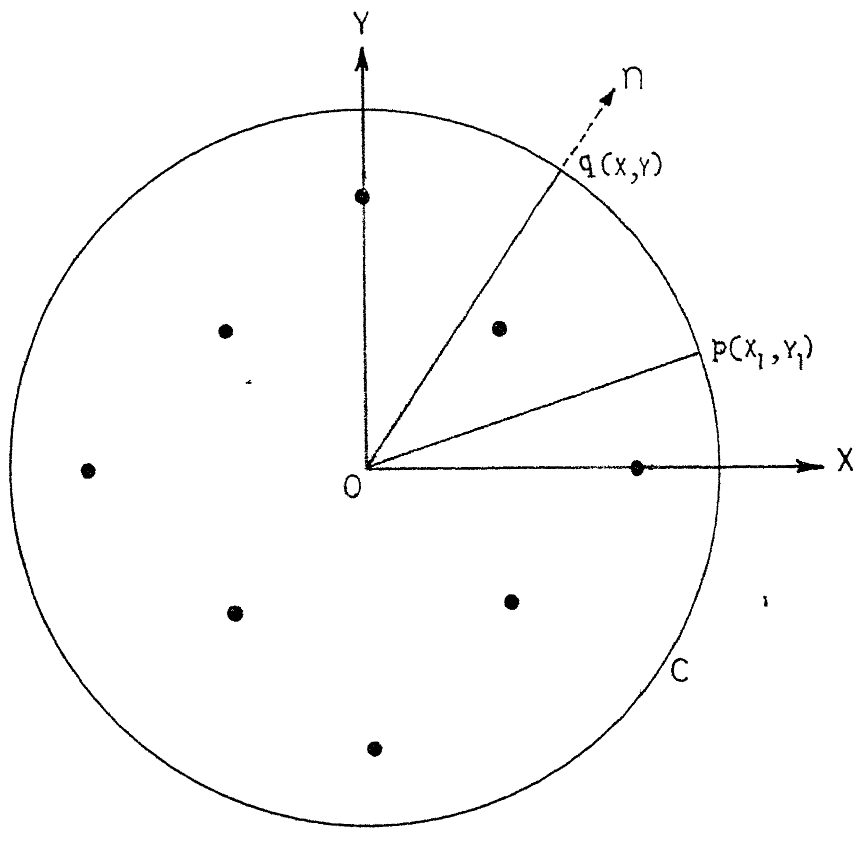

FIG. 7 - $P(X_1,Y_1)$ AND $q(X,Y)$ DENOTE THE FIXED AND VARIABLE POINTS RESPECTIVELY ON THE BOUNDARY AND DOTS INDICATE THE POINTS WHERE THE SOLUTION OF LAPLACE EQUATION IS FOUND.

DIRICHLET PROBLEM FOR CIRCULAR DISC

Values of σ

Table No. 1

$\phi(p) = x$ $\qquad$ N = 8

Gauss Legendre formula		Lobatto formula		Trapezoidal rule	
Analytic values	Computed values	Analytic values	Computed values	Analytic values	Computed values
-.3158361	-.3172925	-.3183098	-.2948808	.3183098	.3873572
-.2555446	-.2552234	-.2928171	-.3342700	.2250790	.2492208
-.0255051	-.0250469	-.0904371	-.0915775	.0000000	.0275257
.2669019	.2655955	.2519430	.2596656	-.2250791	-.2242944
-.3158361	-.3172925	-.3183098	-.2948808	-.3183098	-.3873572
-.2555446	-.2552234	-.2928171	-.3342699	-.2250789	-.2492208
-.0255051	-.0250469	-.0904371	-.0915773	.0000001	-.0275257
.2669019	.2655955	.2519430	.2596658	.2250791	.2242944

Table No. 2

$\phi(p) = x$ $\qquad$ N = 16

-.3181334	-.3156328	-.3183098	-.2972439	.3183098	.3871220
-.3134966	-.3164736	-.3168562	-.3422572	.2940799	.2666482
-.2903678	-.2962494	-.3024826	-.3090697	.2250790	.2351867
-.2288679	-.2349149	-.2527400	-.2596090	.1218118	.1415500
-.1152004	-.1191500	-.1466339	-.1515533	.0000000	.0133837
.0497780	.0419098	.0139361	.0132575	-.1218119	-.1246496
.2016601	.2062528	.1872349	.1915474	-.2250791	-.2433156
.3042346	.3118770	.3023181	.3102732	-.2940799	-.2208678
-.3181334	-.3156328	-.3183098	-.2972439	-.3183098	-.3871220
-.3134966	-.3164736	-.3168562	-.3422572	-.2940799	-.2666482
-.2903678	-.2962494	-.3024826	-.3090696	-.2250789	-.2351867
-.2288679	-.2349149	-.2527400	-.2596088	-.1218117	-.1415500
-.1152004	-.1191500	-.1466339	-.1515531	.0000001	-.0133837
.0497780	.0419098	.0139361	.0132379	.1218120	.1246497
.2016601	.2062528	.1872349	.1916496	.2250791	.2433156
.3042346	.3118770	.3023181	.3102775	.2940800	.2208678

CONTD...

Table No. 3

$\phi(p) = x$ N = 24

Gauss Legendre formula		Lobatto formula		Trapezoidal rule	
Analytic values	Computed values	Analytic values	computed values	Analytic values	Computed values
-.3182734	-.3117388	-.3183098	-.3117960	.3183098	.3762951
-.3173072	-.3163299	-.3180333	-.3178739	.3074637	.3613849
-.3123438	-.3146339	-.3152388	-.3178495	.2756644	.2662459
-.2982584	-.3019636	-.3050229	-.3089800	.2250790	.2316820
-.2687590	-.2730539	-.2805893	-.2852442	.1581548	.1719662
-.2179886	-.2221708	-.2348728	-.2395781	.0823846	.0954383
-.1428086	-.1461405	-.1630991	-.1671394	.0000000	.0092491
-.0452676	-.0470313	-.0658524	-.0685054	-.0823847	-.0796726
.0657301	.0660694	.0484940	.0476715	-.1581548	-.1647020
.1747236	.1773278	.1635387	.1643925	-.2250791	-.2409745
.2626374	.2671695	.2579164	.2590863	-.2756644	-.2977637
.3118861	.3175299	.3113183	.3094737	-.3074637	-.3465382
-.3182734	-.3117391	-.3183098	-.3117957	-.3183098	-.3762950
-.3173072	-.3163299	-.3180333	-.3172016	-.3074636	-.3613849
-.3123438	-.3146339	-.3152388	-.3180130	-.2756643	-.2662459
-.2982584	-.3019636	-.3050229	-.3093356	-.2250789	-.2316820
-.2687590	-.2730539	-.2805893	-.2858895	-.1581548	-.1719662
-.2179886	-.2221708	-.2348728	-.2406841	-.0823845	-.0954383
-.1428086	-.1461405	-.1630991	-.1690604	.0000001	-.0092491
-.0452676	-.0470313	-.0658524	-.0622545	.0823848	.0796726
.0657301	.0660694	.0484940	.0498286	.1581550	.1647020
.1747236	.1773278	.1635387	.1660035	.2250791	.2409745
.2626374	.2671695	.2579164	.2525194	.2756645	.2977637
.3118861	.3175299	.3113183	.3171688	.3074637	.3465382

CONTD...

Table No. 4

$\phi(p) = x$ N = 32

Gauss Legendre formula		Lobatto formula		Trapezoidal rule	
Analytic values	Computed values	Analytic values	Computed values	Analytic values	Computed values
-.3182980	-.3098558	-.3183098	-.3055747	.3183098	.3303176
-.3179847	-.3150675	-.3182240	-.3143002	.3121936	.3363321
-.3163614	-.3167049	-.3173516	-.3176851	.2940799	.2739200
-.3116772	-.3134916	-.3141180	-.3157844	.2646649	.2612319
-.3015736	-.3042255	-.3061819	-.3087486	.2250790	.2301800
-.2833285	-.2864544	-.2906626	-.2938002	.1768434	.1860285
-.2542372	-.2575366	-.2645230	-.2679246	.1218118	.1321951
-.2121339	-.2153007	-.2251247	-.2284706	.0620991	.0716084
-.1560076	-.1587169	-.1709217	-.1738647	.0000000	.0070947
-.0866099	-.0885360	-.1021904	-.1043725	-.0620992	-.0585516
-.0068994	-.0077560	-.0216157	-.0227109	-.1218119	-.1226219
.0778508	.0782605	.0654814	.0657090	-.1768435	-.1826220
.1604217	.1621567	.1514598	.1530985	-.2250791	-.2364192
.2325251	.2354748	.2272948	.2302440	-.2646650	-.2836565
.2860829	.2899635	.2840242	.2879868	-.2940798	-.3167071
.3146512	.3190372	.3144120	.3189278	-.3121936	-.3609205
-.3182980	-.3098560	-.3183098	-.3055743	-.3183098	-.3303176
-.3179847	-.3150677	-.3182240	-.3147006	-.3121935	-.3363321
-.3163614	-.3167049	-.3173516	-.3176851	-.2940799	-.2739200
-.3116772	-.3134916	-.3141180	-.3157844	-.2646648	-.2612319
-.3015736	-.3042255	-.3061819	-.3087486	-.2250789	-.2301800
-.2833285	-.2864545	-.2906626	-.2938002	-.1768433	-.1860285
-.2542373	-.2575366	-.2645230	-.2679246	-.1218117	-.1321951
-.2121339	-.2153007	-.2251247	-.2284707	-.0620990	-.0716084
-.1560076	-.1587169	-.1709217	-.1738647	.0000001	-.0070947
-.0866099	-.0885360	-.1021904	-.1043725	.0620993	.0585516
-.0068994	-.0077561	-.0216157	-.0227109	.1218120	.1226219
.0778508	.0782605	.0654814	.0657089	.1768436	.1826220
.1604217	.1621567	.1514598	.1530986	.2250791	.2364192
.2325251	.2354747	.2272948	.2302440	.2646650	.2836565
.2860829	.2899634	.2840242	.2879868	.2940800	.3167071
.3146512	.3190372	.3144120	.3189278	.3121936	.3309205

VALUES OF V(P)

Table No. 5

$\Phi(p) = x$ $N = 8$

Coordinates of the point P		Analytic value	Computed Values					
X	Y		Gauss-Legendre	Absolute error	Lobatto formula	Absolute error	Trapezoidal rule	Absolute error
.2122	.0000	.2122	.20471242	.00748758	.19676222	.01543778	.19928469	.01291531
.1061	.1061	.1061	.11383563	.00773563	.11716227	.01106227	.10123878	.00486122
.0000	.2122	.0000	-.00172520	.00172520	-.00502260	.00502260	-.00356693	.00356693
-.1061	.1061	-.1061	-.10670039	.00060039	-.10370826	.00239174	-.11203414	.00593414
-.2122	.0000	-.2122	-.21519096	.00299096	-.20688226	.00531774	-.19928469	.01291531
-.1061	-.1061	-.1061	-.10670039	.00060039	-.10370826	.00239174	-.10123878	.00486122
.0000	-.2122	.0000	-.00172520	.00172520	-.00502261	.00502261	.00356692	.00356692
.1061	-.1061	.1061	.11383562	.00773562	.11716227	.01106227	.11203414	.00593414

Table No. 6

$\Phi(p) = x$ $N = 16$

.2122	.0000	.2122	.21688898	.00468898	.21681315	.00461315	.20540890	.00679110
.1061	.1061	.1061	.10938337	.00328337	.10957908	.00347908	.10223900	.00386100
.0000	.2122	.0000	.00096249	.00096249	.00090609	.00090609	-.00254110	.00254110
-.1061	.1061	-.1061	-.10693471	.00083471	-.10687829	.00077829	-.11004487	.00394487
-.2122	.0000	-.2122	-.21412672	.00192672	-.21387942	.00167942	-.20540890	.00679110
-.1061	-.1061	-.1061	-.10693471	.00083471	-.10687829	.00077829	-.10223901	.00386099
.0000	-.2122	.0000	.00096249	.00096249	.00090609	.00090609	.00254109	.00254109
.1061	-.1061	.1061	.10938337	.00328337	.10957908	.00347908	.11004487	.00394487

VALUES OF V(P)

Table No. 7

p) = x

N = 24

dinates of the point P		Analytic value	Computed Values					
X	Y		Gauss-Legendre	Absolute error	Lobatto formula	Absolute error	Trapezoidal rule	Absolute error
122	.0000	.2122	.21611556	.00391556	.21624946	.00404946	.20783088	.00436912
.061	.1061	.1061	.10837860	.00227860	.10847791	.00237791	.10302896	.00307104
000	.2122	.0000	.00046459	.00046459	.00050778	.00050778	-.00189126	.00189126
.061	.1061	-.1061	-.10671389	.00061389	-.10672240	.00062240	-.10875237	.00265237
122	.0000	-.2122	-.21353863	.00133863	-.21353494	.00133494	-.20783089	.00436911
.061	-.1061	-.1061	-.10671389	.00061389	-.10672240	.00062240	-.10302896	.00307104
000	-.2122	.0000	-.00046459	.00046459	.00050777	.00050777	-.00189125	.00189125
.061	-.1061	.1061	.10837861	.00227861	.10847791	.00237791	.10875238	.00265238

Table No. 8

p) = x

N = 32

122	.0000	.2122	.21524597	.00304597	.21533914	.00313914	.20886510	.00333490
.061	.1061	.1061	.10783164	.00173164	.10788752	.00178752	.10354489	.00255511
000	.2122	.0000	.00033870	.00033870	.00035132	.00035132	-.00150127	.00150127
.061	.1061	-.1061	-.10657575	.00047575	-.10658622	.00048622	-.10806035	.00196035
122	.0000	-.2122	-.21321992	.00101992	-.21323570	.00103570	-.20886510	.00333490
061	-.1061	-.1061	-.10657576	.00047576	-.10658622	.00048622	-.10354490	.00255510
000	-.2122	.0000	.00033870	.00033870	.00035132	.00035132	.00150127	.00150127
061	-.1061	.1061	.10783164	.00173164	.10788753	.00178753	.10806036	.00196036

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Values of σ

Table No. 9

$\phi(p) = x^2 - y^2$ $N = 8$

Gauss Legendre formula		Lobatto formula		Trapezoidal rule	
Analytic values	Computed values	Analytic values	Computed values	Analytic values	Computed values
.1963674	.2066433	.2026423	.2037432	.2026423	.2451399
.0585700	.0528288	.1403251	.1412113	.0000000	.0296941
-.2000402	-.2006931	-.1699268	.1623194	-.2026423	-.2328901
.0823042	.0828662	.0512588	.0521325	.0000001	.0389090
.1963674	.2066433	.2026423	.2037433	.2026423	.2451399
.0585700	.0528288	.1403251	.1412112	-.0000001	.0291940
-.2000402	-.2006931	-.1699268	-.1623194	-.2026423	-.2328901
.0823042	.0828662	.0512588	.0521326	.0000002	.0389091

Table No. 10

$\phi(p) = x^2 - y^2$ $N = 16$

.2021931	.1980524	.2026423	.2089591	.2026423	.2809343
.1904783	.1958929	.1989491	.2018768	.1432897	.1348220
.1346116	.1399937	.1633407	.1603467	.0000000	.0183769
.0068799	.0077573	.0528678	.0556040	-.1432897	-.1355681
-.1495577	-.1449214	-.1166362	-.1109961	-.2026423	-.2176495
-.1927309	-.2026100	-.2018654	-.2091632	-.1432897	-.1638121
-.0399751	-.0409803	-.0624147	-.0640259	.0000001	-.0026393
.1675925	.1638732	.1629428	.1690138	.1432898	.1302893
.2021931	.1980523	.2026423	.2063480	.2026423	.2809343
.1904783	.1958929	.1989491	.2018768	.1432896	.1348220
.1346116	.1399937	.1633407	.1603467	-.0000001	.0183769
.0068799	.0077573	.0528678	.0556040	-.1432899	-.1355681
-.1495577	-.1449214	-.1166362	-.1109961	-.2026423	-.2176495
-.1927309	-.2026100	-.2018654	-.2091632	-.1432895	-.1638121
-.0399751	-.0409803	-.0624147	-.0640259	.0000002	-.0026393
.1675925	.1638732	.1629428	.1690138	.1432899	.1302893

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Table No. 11

$\Phi(p) = x^2 - y^2$ N = 24

Gauss Legendre formula		Lobatto formula		Trapezoidal rule	
Analytic values	Computed values	Analytic values	Computed values	Analytic values	Computed values
.2025496	.1942569	.2026423	.1981475	.2026423	.2126745
.2000931	.2018092	.2019384	.2093944	.1754933	.1696109
.1875923	.1913550	.1948598	.1988846	.1013211	.1045782
.1531901	.1573801	.1695136	.1640699	.0000000	.0033570
.0862833	.0894819	.1122792	.1161606	-.1013212	-.0909125
-.0125661	-.0118525	.0180187	.0196171	-.1754934	-.1749624
-.1210651	-.1237294	-.0962370	-.0981734	-.2026423	-.2121803
-.1944457	-.1998378	-.1852961	-.1904776	-.1754933	-.1896943
-.1853605	-.1909805	-.1932356	-.1991994	-.1013210	-.1120438
-.0805289	-.0831163	-.0956626	-.0987727	.0000001	.0007952
.0732714	.0755708	.0634412	.0654971	.1013212	.1142213
.1864496	.1924542	.1850341	.1911548	.1750934	.1700771
.2025491	.1942569	.2026423	.1980573	.2026423	.2126745
.2000931	.2018092	.2019384	.2093944	.1754933	.1696109
.1875923	.1913550	.1948598	.1988846	.1013210	.1045782
.1531901	.1573801	.1695136	.1640699	-.0000001	.0033570
.0862833	.0894819	.1122792	.1161606	-.1013213	-.0909125
-.0125661	-.0118525	.0180187	.0196171	-.1754934	-.1749624
-.1210651	-.1237294	-.0962370	-.0981734	-.2026423	-.2121803
-.1944457	-.1998378	-.1852961	-.1904776	-.1754932	-.1796943
-.1853605	-.1909805	-.1932356	-.1991994	-.1013209	-.1120438
-.0805289	-.0831162	-.0956626	-.0987727	.0000002	.0007952
.0732714	.0755708	.0634412	.0654971	.1013213	.1142213
.1864495	.1924542	.1850341	.1911548	.1754935	.1700771

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Table No. 12

$\phi(p) = x^2 - y^2$ $\qquad$ N = 32

Gauss Legendre formula		Lobatto formula		Trapezoidal rule	
Analytic values	Computed values	Analytic values	Computed values	Analytic values	Computed values
.2026123	.1984526	.2026423	.1995245	.2026423	.2084094
.2018147	.2016295	.2024238	.2073328	.1876170	.1829604
.1976958	.1996434	.2002058	.2021786	.1432897	.1359368
.1859285	.1888126	.1920383	.1948991	.0775478	.0833726
.1611444	.1643703	.1723471	.1756732	.0000000	.0006221
.1184580	.1214368	.1352967	.1385514	-.0775479	-.0679853
.0559041	.0579614	.0772475	.0797409	-.1432897	-.1384695
-.0226390	-.0221666	.0000822	.0010716	-.1872171	-.1885865
-.1052887	-.1068269	-.0857853	-.0868591	-.2026423	-.2095334
-.1726372	-.1761008	-.1608707	-.1640657	-.1876170	-.1972846
-.2024519	-.2070715	-.2007733	-.2054019	-.1432897	-.1532130
-.1783993	-.1828066	-.1854910	-.1901481	-.0775478	-.0838342
-.0997018	-.1023449	-.1108819	-.1138646	.0000001	.0002894
.0136294	.0138521	.0040095	.0039882	.0775479	.0871219
.1247314	.1279090	.1200367	.1231718	.1432898	.1594113
.1933792	.1984265	.1927774	.1979411	.1876171	.1863658
.2026123	.1984522	.2026423	.1990001	.2026423	.2084094
.2018147	.2016296	.2024238	.2073327	.1872170	.1829604
.1976958	.1996434	.2002058	.2021786	.1432896	.1359368
.1859284	.1888125	.1920383	.1948991	.0775477	.0033726
.1611444	.1643703	.1723471	.1756731	-.0000001	.0006221
.1184580	.1214368	.1352967	.1385514	-.0775480	-.0679853
.0559041	.0579614	.0772475	.0797409	-.1432896	-.1384695
-.0226390	-.0221666	.0000822	.0010717	-.1876171	-.1885865
-.1052887	-.1068269	-.0857853	-.0868591	-.2026423	-.2095334
-.1726372	-.1761008	-.1608707	-.1640657	-.1876170	-.1972846
-.2024519	-.2070715	-.2007733	-.2054019	-.1432895	-.1532130
-.1783993	-.1828066	-.1854910	-.1901481	-.0775476	-.0838342
-.0997018	-.1023443	-.1108819	-.1138645	.0000002	.0002894
.0136294	.0138520	.0040095	.0039882	.0775481	.0871219
.1247314	.1279090	.1200367	.1231718	.1432899	.1594113
.1933792	.1984265	.1927774	.1979411	.1876171	.1863658

VALUES OF V (P)

Table No. 13

(p) $= x^2 - y^2$ N = 8

dinates of the point P X	Y	Analytic Value	Computed Values Gauss-Legendre	Absolute error	Lobatto formula	Absolute error	Trapezoidal rule	Absolute error
2122	.0000	.04502883	.02861544	.01641339	.01844908	.02657975	.03023010	.01479873
1061	.1061	.00000000	.00297520	.00297520	.00255872	.00255872	-.01051674	.01051674
0000	.2122	-.04502883	-.05282685	.00779802	-.03993867	.00509016	-.06006643	.01503760
1061	.1061	.00000000	-.00059084	.00059084	-.00407635	.00407635	-.00947558	.00947558
2122	.0000	.04502883	.04513605	.00010722	.03799085	.00703798	.03023010	.01479873
1061	-.1061	.00000000	-.00059084	.00059084	-.00407635	.00407635	-.01051674	.01051674
0000	-.2122	-.04502883	-.05282685	.00779802	.00255872	.04247011	-.06006643	.01503760
1061	-.1061	.00000000	.00297520	.00297520	-.03993867	.03993867	-.00947558	.00947558

Table No. 14

(p) $= x^2 - y^2$ N = 16

X	Y	Analytic Value	Gauss-Legendre	Absolute error	Lobatto formula	Absolute error	Trapezoidal rule	Absolute error
122	.0000	.04502883	.04585153	.00082270	.04546563	.00043680	.03848173	.00654710
061	.1061	.00000000	-.00008027	.00008027	-.00012891	.00012891	-.00473142	.00473142
000	.2122	-.04502883	-.04696237	.00193354	-.04749385	.00246502	-.05038008	.00535125
061	.1061	.00000000	-.00052419	.00052419	-.00062139	.00062139	-.00332281	.00332281
122	.0000	.04502883	.04550470	.00047587	.04528405	.00025522	.03848173	.00654710
061	-.1061	.00000000	-.00052419	.00052419	-.00062139	.00062139	-.00473142	.00473142
000	-.2122	-.04502883	-.04696237	.00193354	-.04749386	.00246503	-.05038008	.00535125
061	-.1061	.00000000	-.00008027	.00008027	-.00012892	.00012892	-.00332281	.00332281

VALUES OF V(P)

Table No. 15

$\cdot(p) = x^2 - y^2$ N = 24

ordinates of the point P: X	Y	Analytic value	Computed Values: Gauss-Legendre	Absolute error	Lobatto formula	Absolute error	Trapezoidal rule	Absolute error
.2122	.0000	.04502883	.04641649	.00138766	.04643008	.00140125	.04106105	.00396778
.1061	.1061	.00000000	-.00002336	.00002336	-.00002772	.00002772	-.00310132	.00310132
.0000	.2122	-.04502883	-.04640837	.00137954	-.04646284	.00143401	-.04836747	.00333864
1061	.1061	.00000000	-.00035832	.00035832	-.00038384	.00038384	-.00204826	.00204826
2122	.0000	.04502883	.04539116	.00036233	.04536546	.00033663	.04106105	.00396778
1061	-.1061	.00000000	-.00035832	.00035832	-.00038385	.00038385	-.00310133	.00310133
0000	-.2122	-.04502883	-.04640837	.00137954	-.04646284	.00143401	-.04836748	.00333865
1061	-.1061	.00000000	-.00002336	.00002336	-.00002772	.00002772	-.00204826	.00204826

Table No. 16

$\cdot(p) = x^2 - y^2$ N = 32

2122	.0000	.04502883	.04619621	.00116738	.04622509	.00119626	.04211479	.00291404
1061	.1061	.00000000	-.00001580	.00001580	-.00001739	.00001739	-.00235066	.00235066
0000	.2122	-.04502883	-.04613385	.00110502	-.04616451	.00113568	-.04746068	.00243185
1061	.1061	.00000000	-.00027095	.00027095	-.00028247	.00028247	-.00151206	.00151206
2122	.0000	.04502883	.04531281	.00028398	.04531043	.00028160	.04211479	.00291404
1061	-.1061	.00000000	-.00027095	.00027095	-.00028248	.00028243	-.00235066	.00235066
0000	-.2122	-.04502883	-.04613386	.00110503	-.04616452	.00113569	-.04746068	.00243185
1061	-.1061	.00000000	-.00001580	.00001580	-.00001739	.00001739	-.00151207	.00151207

CHAPTER 4

DIRICHLET PROBLEM FOR A CIRCULAR DISC BY SECOND METHOD

In the last chapter we have seen the usefulness of the First Method in solving Dirichlet Problem. In this chapter the application of the Second Method is discussed for the same problems under similar boundary conditions to make a comparative study between the two methods.

As mentioned earlier the pair of integral equations to be solved in this case are as follows.

$$g(p) = \frac{1}{2\pi} \int_{0}^{2} \mu(q) \log'|q-p| dq + \frac{1}{2}\mu(p) \qquad \ldots (70)$$

and

$$W(P) = \frac{1}{2\pi} \int_{0}^{2} \mu(q) \log'|q-P| dq \qquad \ldots (71)$$

where the length of the perimeter of the circle is 2; other symbols carry their usual meaning. For the kernel $\log'|q-p|$, we have made use of (45) here i.e.,

$$\log'|q-p| = \frac{(x-x_1)\cos\beta + (y-y_1)\sin\beta}{(x-x_1)^2 + (y-y_1)^2}$$

where (x,y) and (x_1,y_1) are the coordinates of the points q and p respectively and β is the angle subtended by the outward normal at q with x-axis (fig.7,pp. 50). If α be the inclination of the radius vector of the point p, then the points q and

p in this case can be replaced by $\left(\frac{\cos\beta}{\pi}, \frac{\sin\beta}{\pi} \right)$ and $\left(\frac{\cos\alpha}{\pi}, \frac{\sin\alpha}{\pi} \right)$ respectively. Substituting these values in the above equation, we find that

$$\log' |q-p| = \frac{\pi}{2} \qquad \ldots (72)$$

Observing that in case of the circle of radius $1/\pi$, which we are considering, $ds = \frac{1}{\pi} d\theta$, thus (70) reduces to the following form

$$g(p) = \frac{1}{4\pi} \int_0^{2\pi} \mu \left(\frac{\cos\theta}{\pi}, \frac{\sin\theta}{\pi} \right) d\theta + \frac{1}{2} \mu(p) \qquad \ldots (73)$$

We change the limits of the above integral from $(0,2\pi)$ to $(-1,+1)$ by putting

$$\theta = \pi\Phi + \pi, \quad d\theta = \pi d\Phi \qquad \ldots (74)$$

thus,

$$g(p) = \frac{1}{4} \int_{-1}^{1} \mu\left(-\frac{1}{\pi}\cos\pi\Phi, -\frac{1}{\pi}\sin\pi\Phi\right) d\Phi + \frac{1}{2}\mu(p).$$

Replacing p by p_i and using Gauss-Legendre quadrature formula in the above equation, we get

$$g(p_i) = \frac{1}{4} \sum_{k=1}^{N} w_k \, \mu\left(-\frac{\cos\pi\Phi_k}{\pi}, -\frac{\sin\pi\Phi_k}{\pi}\right) + \frac{1}{2}\mu(p_i) \qquad \ldots (75)$$

Denoting $\mu\left(-\frac{\cos\pi\Phi_k}{\pi}, -\frac{\sin\pi\Phi_k}{\pi}\right)$ by μ_k and taking these nodal points as the fixed points i.e.,

$$p_i = \left(-\frac{\cos\pi\Phi_i}{\pi}, -\frac{\sin\pi\Phi_i}{\pi}\right), \; i = 1,2,\ldots,N$$

equation (75) can be written as

$$4.g(p_i) = \sum_{k\neq i}^{N} w_k \mu_k + (w_i+2)\ \mu_i .$$

In the first case where g(p) = x, we have

$$-\frac{4}{\pi} \cos \pi \Phi_i = \sum_{k\neq i}^{N} w_k \mu_k + (w_i+2)\mu_i \qquad \ldots (76)$$

which is a system of N-linear simultaneous equations, with the following coefficient matrix.

$$G_n = \begin{bmatrix} (w_1+2) & w_2 & \cdots\cdots & w_N \\ w_1 & (w_2+2) & \cdots\cdots & w_N \\ \cdot\cdot & \cdot\cdot & & \cdot\cdot \\ \cdot\cdot & \cdot\cdot & & \cdot\cdot \\ w_1 & w_2 & \cdots\cdots & (w_N+2) \end{bmatrix}$$

We have in fact already proved in Chapter 2 that the matrix G_N is non-singular, in general, but the same result can be proved in this case with the help of the following theorem concerning matrices. It is important to point out here that w_i, i = 1,2,.....N are the weights of the Gauss Legendre quadrature formula and hence are positive and also

$$\sum_{i=1}^{N} w_i = 2 \qquad \ldots (77)$$

Following is the statement of the theorem.

Let $A = \left[a_{ij}\right]$, be an arbitrary square matrix of order N.

If

$$\left|a_{ii}\right| > \sum_{j\neq i}^{N} \left|a_{ij}\right| , \quad (i = 1,2,\ldots.N) \qquad \ldots (78)$$

then A is non-singular.

Suppose that A is singular and let $\vec{y} = (y_1, y_2, \ldots., y_N)$ be a non zero vector satisfying the equation $A\vec{y} = 0$. Nothe that such a vector would exist for a singular matrix A. Let $y_k = \max_i \; y_i$, then taking the kth row,

$$\left|a_{kk}\right| \left|y_k\right| = \left|a_{kk} y_k\right| = \left|\sum_{j\neq k}^{N} a_{kj} \; y_j\right|$$

$$\leq \sum_{j\neq k}^{N} \left|a_{kj}\right| \left|y_j\right| \leq \left|y_k\right| \sum_{j\neq k}^{N} \left|a_{kj}\right|$$

$$< \left|a_{kk}\right| \left|y_k\right| \text{ , from (78) .}$$

which is impossible and therefore A is non-singular.

Now from(77),

$$\sum_{k=1}^{N} w_k = \sum_{k\neq i}^{N} w_k + w_i = 2$$

Therefore $(w_i+2) > \sum_{k\neq i}^{N} w_k$, since $w_i > 0$, $i=1,2,\ldots..N$. Consequently from the theorem already proved G_N is non-singular and so the solution of (76) exists. Crout's method was used to compute the values of μ's. It may be noted that in this case also, we find the analytic values of μ's , for different N, by using (23). In particular, when $g(p) = x$

$$\mu\left(\frac{1}{\pi} , \theta\right) = \frac{\cos \theta}{\pi^2} \qquad \ldots (79)$$

and when $g(p) = x^2 - y^2$,

$$\mu\left(\frac{1}{\pi} ,\theta\right) = \frac{\cos 2\theta}{\pi^2} \qquad \ldots (80)$$

Analytic and computed values of μ's for N = 8,16,24 and 32 for Gauss-Legendre quadrature formula are shown in Table Nos.17-20, pps. 70-72 . The maximum error for N = 32 is .00007 % Then to find W(P) , we take the coordinates of the points P and q as (R,η) and $(1/\pi,\theta)$ respectively. Equation (71) on simplification reduces to the following equation.

$$W(R,\eta) = \frac{1}{2\pi^2} \int_0^{2\pi} \mu\left(\frac{\cos \theta}{\pi} , \frac{\sin \theta}{\pi} \right) \left\{ \frac{1}{\pi} - R \cos(\theta-\eta)\right\}$$

$$\left\{ \frac{1}{\pi^2} + R^2 - \frac{2R}{\pi} \cos(\theta-\eta) \right\}^{-1} d\theta \qquad \ldots (81)$$

After changing the limits of the above integral to (-1,+1), using (74) and then apply Gauss-Legendre quadrature formula, to obtain

$$W(P) = \frac{1}{2\pi} \sum_{k=1}^{N} w_k \mu_k \left\{ \frac{1}{\pi} + R \cos(\pi \Phi_k - \eta) \right\}$$

$$\left\{ \frac{1}{\pi^2} + R^2 + \frac{2R}{\pi} \cos(\pi \Phi_k - \eta)\right\}^{-1} \qquad \ldots (82)$$

Thus substituting the values of μ's obtained from (76) and w_k, Φ_k from the tables [52], we find W(P) at all those eight points, mentioned in the last chapter and shown in fig.7,pp. 50 .

Computed and analytic values of W(P) at these points are given in Table Nos. 21-24,pps. 73, 74 . The maximum error for $N = 32$ at any of these points is .08 %.

For Lobatto quadrature formula, we have to change weights and abscissas only in (76) and (82), according to the tables [52]. The analytic and computed values of μ's, for values of $N = 8,16,24,32$ are given in Table Nos. 17-20,pps. 70-72 . Here also the maximum error for $N = 32$ is .00005 % . Similarly the values of W(P) obtained for different values of N along with the absolute error are given in Table Nos. 21-24, pps. 73, 74 . The maximum error for $N = 32$ is .09 % .

For the trapezoidal rule we proceed as follows. Applying this rule in the form given in (57) into (73), we get

$$g(p) = \frac{1}{2N} \sum_{k=1}^{N} \mu_k + \frac{1}{2} \mu(p) ,$$

where

$$\mu_k = \mu \left(\frac{1}{\pi} \cos \frac{2\pi(k-1)}{N} , \frac{1}{\pi} \sin \frac{2\pi(k-1)}{N} \right) .$$

Replacing in the above equation p by p_i, where

$$p_i = \left(\frac{1}{\pi} \cos \frac{2\pi(i-1)}{N} , \frac{1}{\pi} \sin \frac{2\pi(i-1)}{N} \right); \ i=1,2,\ldots,N$$

Since we are considering the first case where $g(p) = x$, hence

$$\frac{2N}{\pi} \cos \left(\frac{2\pi(i-1)}{N} \right) = \sum_{k \neq i} \mu_k + (N+1)\mu_i \qquad \ldots (83)$$

The above system of N linear simultaneous equations has the

following coefficient matrix,

$$T_N = \begin{bmatrix} (N+1) & 1 & \ldots\ldots\ldots & 1 \\ 1 & (N+1) & \ldots\ldots\ldots & 1 \\ \cdot\cdot & \cdot\cdot & & \cdot\cdot \\ \cdot\cdot & \cdot\cdot & & \cdot\cdot \\ 1 & 1 & \ldots\ldots & (N+1) \end{bmatrix}$$

which is non-singular for the reasons given earlier in case of the matrix G_N. Therefore we can solve (83) to get μ's. These values along with their analytical values are given in Table Nos.17-20, pps. 70-72 , for the values of N mentioned above. The maximum error for N = 32 is practically nil. Then we applied trapezoidal rule to (81), which on simplification reduces to the following form.

$$W(P) = \frac{1}{\pi N} \sum_{k=1}^{N} \mu_k \left\{ \frac{1}{\pi} - R \cos \left(\frac{2\pi(k-1)}{N} - \eta \right) \right\}$$

$$\left\{ \frac{1}{\pi^2} + R^2 - \frac{2R}{\pi} \cos \left(\frac{2\pi(k-1)}{N} - \eta \right) \right\}^{-1} \qquad \ldots (84)$$

Substituting the values of μ's obtained earlier we finally get W(P) at all those eight points. These values along with their analytical values appear in Table Nos. 21-24, pps. 73, 74 , The maximum error at any point for N = 32 is almost zero.

For the second case where $g(p) = x^2 - y^2$, the procedure is the same except for few changes in equations (76) and (83). Here too all the three quadrature formulae were used. The

values of μ's obtained are given in Table Nos. 25-28,pps. 75-77. The maximum error for N = 32 by Gauss-Legendre quadrature formula is .0003 % , by Lobatto quadrature formula .12 % and by trapezoidal rule .00002 % . Similarly the values of W(P) are again given in Table Nos. 29-32,pps. 78, 79 . The maximum error at any point for N = 32 by Gauss-Legendre quadrature formula is .14 %, Lobatto quadrature formula .18 % , and by trapezoidal rule .0014 % . Computational work was done on the Computer IBM/7044 and programmes for one of the problems where g(p) = $x^2 - y^2$, using Gauss-Legendre quadrature formula and trapezoidal rule are given in Appendix II.

Looking at the results of the last and the present chapter, it may be seen that in the First Method, Gauss-Legendre quadrature formula provides better results while in the Second Method, it is the trapezoidal rule which gives pretty accurate results. The maximum error is about one in a lakh. The reason for this is as follows. The integral involved in the Second Method to which we applied trapezoidal rule is $\int_0^{2\pi} \mu(\cos\theta/\pi, \sin\theta/\pi)\,d\theta$. The integrand is a periodic function of $\sin\theta$ and $\cos\theta$, of period 2π. Consequently, since $\mu(0) = \mu(2\pi)$, the formula (57) was applied. Now consider the error

$$E_{T_N}(f) = \frac{p}{N}\sum_{k=1}^{N} f\left(\frac{k-1}{N}\,p\right) - \int_0^p f(x)\,dx, \text{ in this formula.}$$

It can be easily verified that

$$E_{T_N}\left(e^{i2\pi jx/p} \right) = \begin{cases} p\ , & j \neq 0,\ N/j \\ 0, & \text{otherwise} \end{cases}\ , \quad i = \sqrt{-1}$$

This means that the trapezoidal rule (T_N) is exact for 2N periodic functions 1, sin x, cos x,...., sin(N-1)x, cos(N-1)x, sin Nx. The reason why Gauss-Legendre quadrature formula and Lobatto formula give error is that μ has derivatives of all orders.

VALUES OF μ

Table No. 17

$g(p) = -x$ $\qquad$ $N = 8$

Gauss-Legendre formula		Lobatto formula		Trapezoidal rule	
Analytic values	Computed values	Analytic values	Computed values	Analytic values	Computed values
.6316722	.6316722	.6366197	.6366197	.6366196	.6366197
.5110893	.5110892	.5856342	.5856341	.4501580	.4501581
.0510102	.0510100	.1808741	.1808740	-.0000001	.0000000
-.5338038	-.5338038	-.5038860	-.5038861	-.4501582	-.4501581
.6316722	.6316722	.6366197	.6366197	-.6366197	-.6366197
.5110893	.5110892	.5856342	.5856341	-.4501579	-.4501581
.0510102	.0510100	.1808741	.1808740	.0000003	.0000000
-.5338038	-.5338038	-.5038860	-.5038861	.4501583	.4501581

Table No. 18

$g(p) = x$ $\qquad$ $N = 16$

.6362668	.6362668	.6366197	.6366197	.6366192	.6366197
.6269933	.6269933	.6337124	.6337125	.5881599	.5881599
.5807357	.5807357	.6049653	.6049653	.4501580	.4501581
.4577359	.4577358	.5054801	.5054800	.2436237	.2436238
.2304009	.2304009	.2932679	.2932678	-.0000001	.0000000
-.0995560	-.0995561	-.0278722	-.0278722	-.2436239	-.2436238
-.4033202	-.4033204	-.3744698	-.3744699	-.4501582	-.4501581
-.6084693	-.6084693	-.6046363	-.6046364	-.5881599	-.5881599
.6362668	.6362668	.6366197	.6366197	-.6366197	-.6366197
.6269933	.6269933	.6337124	.6337125	-.5881598	-.5881599
.5807357	.5807357	.6049653	.6049653	-.4501579	-.4501581
.4577359	.4577358	.5054801	.5054800	-.2436235	-.2436238
.2304009	.2304009	.2932679	.2932678	.0000003	.0000000
-.0995560	-.0995560	-.0278722	-.0278722	.2436241	.2436238
-.4033202	-.4033204	-.3744698	-.3744699	.4501583	.4501581
-.6084693	-.6084693	-.6046363	-.6046364	.5881600	.5881599

VALUES OF μ

Table No. 19

$g(p) = x$ N = 24

Gauss-Legendre formula		Lobatto formula		Trapezoidal rule	
Analytic values	Computed values	Analytic values	Computed values	Analytic values	Computed values
.6365469	.6365470	.6366197	.6366197	.6366196	.6366197
.6346144	.6346144	.6360666	.6360666	.6149272	.6149274
.6246876	.6246876	.6304777	.6304777	.5513288	.5513289
.5965169	.5965169	.6100458	.6100458	.4501580	.4501581
.5375180	.5375179	.5611787	.5611787	.3183097	.3183098
.4359773	.4359772	.4697457	.4697456	.1647692	.1647693
.2856172	.2856170	.2361982	.3261981	-.0000001	.0000000
.0905351	.0905349	.1317048	.1317046	-.1747692	-.1647693
-.1314602	-.1414603	-.0969879	-.0969880	-.3183099	-.3183098
-.3494473	-.3494474	-.3270775	-.3270776	-.4501582	-.4501581
-.5252749	-.5252748	-.5158328	-.5158329	-.5513289	-.5513288
-.6237722	-.6237723	-.6226366	-.6226367	-.6149274	-.6149274
.6365469	.6365470	.6366197	.6366197	-.6366197	-.6366197
.6346614	.6346144	.6360666	.6360666	-.6149273	-.6149274
.6246876	.6246876	.6304777	.6304777	-.5513287	-.5513289
.5965169	.5965169	.6100458	.6100458	-.4501579	-.4501581
.5375180	.5375179	.5611787	.5611787	-.3183098	-.3183099
.4359773	.4359772	.4697457	.4697456	-.1647690	-.1647693
.2856172	.2856170	.3261982	.3261981	.0000003	.0000000
.0905351	.0905349	.1317048	.1317046	.1647696	.1647692
-.1314602	-.1314603	-.0969879	-.0969880	.3183101	.3183098
-.3494473	-.3494474	-.3270775	-.3270776	.4501583	.4501581
-.5252749	-.5252748	-.5158328	-.5158329	.5513290	.5513288
-.6237722	-.6237723	-.6226366	-.6226367	.6149275	.6149274

VALUES OF μ

Table No. 20

$g(p) = x$ $N = 32$

Gauss-Legendre formula		Lobatto formula		Trapezoidal rule	
Analytic values	Computed values	Analytic values	Computed values	Analytic values	Computed values
.6365961	.6365962	.6366197	.6366197	.6366196	.6366197
.6359694	.6359694	.6364480	.6364481	.6243872	.6243873
.6327228	.6327228	.6347032	.6347032	.5881599	.5881599
.6233544	.6233544	.6282361	.6282361	.5293298	.5293299
.6031473	.6031473	.6123638	.6123638	.4501580	.4501581
.5666571	.5666570	.5813252	.5813252	.3536868	.3536869
.5084745	.5084745	.5290461	.5290461	.2436237	.2436238
.4242679	.4242679	.4502494	.4502493	.1241982	.1241983
.3120153	.3120152	.3418434	.3418433	-.0000001	.0000000
.1732198	.1732196	.2043808	.2043807	-.1241984	-.1241983
.0137988	.0137987	.0432314	.0432312	-.2436239	-.2436238
-.1557016	-.1557018	-.1309627	-.1309628	-.3536870	-.3536869
-.3208434	-.3208434	-.3029197	-.3029198	-.4501582	-.4501581
-.4650502	-.4650503	-.4545897	-.4545898	-.5293300	-.5293299
-.5721658	-.5721659	-.5680484	-.5680485	-.5881599	-.5881599
-.6293024	-.6293025	-.6288241	-.6288241	-.6243872	-.6243872
.6365961	.6365962	.6366197	.6366197	-.6366197	-.6366197
.6359694	.6359694	.6364480	.6364481	-.6243871	-.6243872
.6327228	.6327228	.6347032	.6347032	-.5881598	-.5881599
.6233544	.6233544	.6282361	.6282361	-.5293297	-.5293300
.6031473	.6031473	.6123638	.6123638	-.4501579	-.4501581
.5666571	.5666570	.5813252	.5813252	-.3536867	-.3536870
.5084745	.5084745	.5290461	.5290461	-.2436235	-.2436238
.4242679	.4242679	.4502494	.4502493	-.1241980	-.1241983
.3120153	.3120152	.3418434	.3418433	.0000003	.0000000
.1732198	.1732196	.2043808	.2043807	.1241986	.1241983
.0137988	.0137987	.0432314	.0432312	.2436241	.2436238
-.1557016	-.1557018	-.1309627	-.1309628	.3536872	.3536869
-.3208434	-.3208434	-.3029197	-.3029198	.4501583	.4501581
-.4650502	-.4650503	-.4545897	-.4545898	.5293301	.5293299
-.5721658	-.5721659	-.5680484	-.5680485	.5881600	.5881599
-.6293024	-.6293025	-.6288241	-.6288241	.6243873	.6243872

VALUES OF W(P)

Table No. 21

p) = x N = 8

dinates of the point P X	Y	Analytic value	Computed Values: Gauss-Legendre	Absolute error	Lobatto formula	Absolute error	Trapezoidal rule	Absolute error
122	.0000	.2122	.21296054	.00076054	.20713304	.00506696	.24019559	.02799559
061	.1061	.1061	.10580904	.00029096	.10133982	.00476018	.10752626	.00142626
000	.2122	.0000	.01403386	.01403386	.02780355	.02780355	.00000000	.00000000
061	.1061	-.1061	-.11574287	.00964287	-.12467177	.01857177	-.10752627	.00142627
122	.0000	-.2122	-.14290320	.06929680	-.12386503	.08833497	-.24019560	.02799560
061	-.1061	-.1061	-.11574286	.00964286	-.12467178	.01857178	-.10752627	.00142627
000	-.2122	.0000	.01403384	.01403384	.02780353	.02780353	-.00000001	.00000001
061	-.1061	.1061	.10580903	.00029097	.10133981	.00476019	.10752626	.00142626

Table No. 22

p) = x N = 16

dinates of the point P X	Y	Analytic value	Computed Values: Gauss-Legendre	Absolute error	Lobatto formula	Absolute error	Trapezoidal rule	Absolute error
122	.0000	.2122	.21220640	.00000640	.21218080	.00001920	.21325105	.00105105
061	.1061	.1061	.10609922	.00000078	.10607733	.00002267	.10610346	.00000346
000	.2122	.0000	-.00197020	.00197020	-.00083002	.00083002	.00000000	.00000000
061	.1061	-.1061	-.10603029	.00006971	-.10583412	.00026588	-.10610346	.00000346
22	.0000	-.2122	-.20249697	.00970303	-.19959964	.01260036	-.21325106	.00105106
61	-.1061	-.1061	-.10603029	.00006971	-.10583412	.00026588	-.10610347	.00000347
00	-.2122	.0000	-.00197022	.00197022	-.00083003	.00083003	-.00000001	.00000001
61	-.1061	.1061	.10609922	.00000078	.10607732	.00002268	.10610346	.00000346

VALUES OF W(P)

Table No. 23

(p) = x N = 24

dinates of the point P X	Y	Analytic value	Computed Values Gauss-Legendre	Absolute error	Lobatto formula	Absolute error	Trapezoidal rule	Absolute error
2122	.0000	.2122	.21220000	.00000000	.21219993	.00000007	.21224091	.00004091
1061	.1061	.1061	.10609999	.00000001	.10609987	.00000013	.10609999	.00000001
0000	.2122	.0000	.00004148	.00004148	-.00018860	.00018860	-.00000000	.00000000
1061	.1061	-.1061	-.10609809	.00000191	-.10610034	.00000034	-.10610000	.00000000
2122	.0000	-.2122	-.21094650	.00125350	.21057059	.00162941	-.21224091	.00004091
1061	-.1061	-.1061	-.10609809	.00000191	-.10610033	.00000033	-.10610001	.00000001
0000	-.2122	.0000	.00004146	.00004146	-.00018862	.00018862	-.00000001	.00000001
1061	-.1061	.1061	.10609997	.00000003	.10609986	.00000014	.10609999	.00000001

Table No. 24

(p) = x N = 32

X	Y	Analytic value	Gauss-Legendre	Absolute error	Lobatto formula	Absolute error	Trapezoidal rule	Absolute error
2122	.0000	.2122	.21219996	.00000004	.21219996	.00000004	.21220157	.00000157
1061	.1061	.1061	.10609998	.00000002	.10609998	.00000002	.10609998	.00000002
0000	.2122	.0000	.00001407	.00001407	.00002187	.00002187	.00000000	.00000000
1061	.1061	-.1061	-.10610008	.00000008	-.10610010	.00000010	-.10609998	.00000002
2122	.0000	-.2122	-.21203985	.00016015	-.21199200	.00020800	-.21220156	.00000156
1061	-.1061	-.1061	-.10610009	.00000009	-.10610010	.00000010	-.10610000	.00000000
0000	-.2122	.0000	.00001406	.00001406	.00002186	.00002186	.00000000	.00000000
1061	-.1061	.1061	.10609998	.00000002	.10609997	.00000003	.10609998	.00000002

VALUES OF μ

Table No. 25

$g(p) = x^2 - y^2$ $N = 8$

Gauss-Legendre formula		Lobatto formula		Trapezoidal rule	
Analytic values	Computed values	Analytic values	Computed values	Analytic values	Computed values
.1963674	.1963678	.2026423	.2026520	.2026423	.2026423
.0585700	.0585702	.1403251	.1403347	.0000000	.0000000
-.2000402	-.2000399	-.1699268	-.1699172	-.2026423	-.2026423
.0823042	.0823046	.0512582	.0512685	.0000001	.0000000
.1963674	.1963678	.2026423	.2026520	.2026423	.2026423
.0585700	.0585702	.1403251	.1403347	-.0000001	.0000000
-.2000402	-.2000399	-.1699268	-.1699172	-.2026423	-.2026423
.0823042	.0823046	.0512588	.0512685	.0000002	.0000000

Table No. 26

$g(p) = x^2 - y^2$ $N = 16$

.2021931	.2021931	.2026423	.2026423	.2026423	.2026423
.1904783	.1904783	.1989491	.1989492	.1432897	.1432897
.1346116	.1346115	.1633407	.1633407	.0000000	.0000000
.0068799	.0068797	.0528678	.0528677	-.1432897	-.1432897
-.1495577	-.1495577	-.1166362	-.1166363	-.2026423	-.2026423
-.1927309	-.1927327	-.2018654	-.2018655	-.1432897	-.1432897
-.0399751	-.0399749	-.0624147	-.0624146	.0000001	.0000000
.1675925	.1675926	.1629428	.1629428	.1432898	.1432897
.2021931	.2021931	.2026423	.2026423	.2026423	.2026423
.1904783	.1904783	.1989491	.1989492	.1432896	.1432897
.1346116	.1346115	.1633407	.1633407	-.0000001	.0000000
.0068799	.0068797	.0528678	.0528677	-.1432899	-.1432897
-.1495577	-.1495577	-.1166362	.1166363	-.2026423	-.2026423
-.1927309	-.1927327	-.2018654	-.2018655	-.1432895	-.1432897
-.0399751	-.0399749	-.0624147	-.0624146	.0000002	.0000000
.1675925	.1675926	.1629428	.1629428	.1432899	.1432897

VALUES OF μ

Table No. 27

$g(p) = x^2 - y^2$ $N = 24$

Gauss-Legendre formula		Lobatto formula		Trapezoidal rule	
Analytic values	Computed values	Analytic values	Computed values	Analytic values	Computed values
.2025496	.2025497	.2026423	.2026423	.2026423	.2026423
.2000931	.2000931	.2019384	.2019384	.1754933	.1754934
.1875923	.1875923	.1948598	.1948598	.1013211	.1013211
.1531901	.1531901	.1695136	.1695136	.0000000	.0000000
.0862833	.0862832	.1122792	.1122791	-.1013212	-.1013211
-.0125661	-.0125662	.0180187	.0180185	-.1754934	-.1754934
-.1210651	-.1210652	-.0962370	-.0962371	-.2026423	-.2026423
-.1944457	-.1944457	-.1852961	-.1852962	-.1754933	-.1754934
-.1853605	-.1853605	-.1932356	-.1932356	-.1013210	-.1013211
-.0805289	-.0805288	-.0956626	-.0956625	.0000001	.0000000
.0732714	.0732712	.0634412	.0634412	.1013212	.1013211
.1864495	.1864495	.1850341	.1850341	.1754934	.1754934
.2025496	.2025497	.2026423	.2026423	.2026423	.2026423
.2000931	.2000931	.2019384	.2019384	.1754933	.1754934
.1875923	.1875923	.1948598	.1948598	.1013210	.1013212
.1531901	.1531901	.1695136	.1695136	-.0000001	.0000000
.0862833	.0862832	.1122792	.1122791	-.1013213	-.1013211
-.0125661	-.0125662	.0180187	.0180185	-.1754934	-.1754934
-.1210651	-.1210652	-.0962370	-.0962371	-.2026423	-.2026423
-.1944457	-.1944457	-.1852961	-.1852962	-.1754932	-.1754934
-.1853605	-.1853605	-.1932356	-.1932356	-.1013208	-.1013212
-.0805289	-.0805288	-.0956626	-.0956625	.0000002	.0000000
.0732714	.0732712	.0634412	.0634412	.1013213	.1013211
.1864495	.1864495	.1850341	.1850341	.1754935	.1754934

VALUES OF μ

Table No. 28

$g(p) = -x^2 - y^2$ N = 32

Gauss-Legendre formula		Lobatto formula		Trapezoidal rule	
Analytic values	Computed values	Analytic values	Computed values	Analytic values	Computed values
.2026123	.2026124	.2026423	.2026423	.2026423	.2026423
.2018147	.2018148	.2024238	.2024238	.1872170	.1872171
.1976958	.1976958	.2002058	.2002058	.1432897	.1432897
.1859284	.1859284	.1920383	.1920383	.0775478	.0775478
.1611444	.1611443	.1723471	.1723471	.0000000	.0000000
.1184580	.1184578	.1352967	.1352966	-.0775478	-.0775478
.0559041	.0559039	.0772475	.0772474	-.1432897	-.1432897
-.0226390	-.0226390	.0000822	.0000821	-.1872171	-.1872171
-.1052887	-.1052888	-.0857853	-.0857855	-.2026423	-.2026423
-.1726372	-.1726373	-.1608707	-.1608708	-.1872170	-.1872171
-.2024519	-.2024519	-.2007733	-.2007734	-.1432897	-.1432897
-.1783993	-.1783993	-.1854910	-.1854911	-.0775477	-.0775478
-.0997018	-.0997018	-.1108819	-.1108819	.0000001	.0000000
-.0136294	-.0136294	.0040095	.0040096	.0775479	.0775478
.1247314	.1247314	.1200367	.1200367	.1432898	.1432897
.1933792	.1933792	.1927774	.1927774	.1872171	.1872171
.2026123	.2026124	.2026423	.2026423	.2026423	.2026423
.2018147	.2018148	.2024238	.2024238	.1872170	.1872171
.1976958	.1976958	.2002058	.2002058	.1432896	.1432897
.1859284	.1859284	.1920383	.1920383	.0775477	.0775478
.1611444	.1611443	.1723471	.1723471	-.0000001	.0000000
.1184580	.1184578	.1352967	.1352966	-.0775480	-.0775478
.0559041	.0559039	.0772475	.0772474	-.1432899	-.1432897
-.0226390	-.02263908	.0000822	.0000821	-.1872171	-.1872171
-.1052887	-.1052888	-.0857853	-.0857855	-.2026423	-.2026423
-.1726372	-.1726373	-.1608707	-.1608708	-.1872170	-.1872171
-.2024519	-.2024519	-.2007733	-.2007734	-.1432895	-.1432897
-.1783993	-.1783993	-.1854910	-.1854911	-.0775476	-.0775478
-.0997018	-.0997018	-.1108819	-.1108819	.0000002	.0000000
.0136294	.0136294	.0040095	.0040096	.0775480	.0775478
.1247314	.1247314	.1200367	.1200367	.1432899	.1432897
.1933792	.1933792	.1927774	.1927774	.1872171	.1872171

VALUES OF W(P)

Table No. 29

N = 8

p) = $x^2 - y^2$

dinates of the point P		Analytic value	Computed Values					
X	Y		Gauss-Legendre	Absolute error	Lobatto formula	Absolute error	Trapezoidal rule	Absolute error
122	.0000	.04502883	.04532990	.00030107	.04302296	.00200587	.05611115	.01108232
061	.1061	.00000000	.00112405	.00112405	-.00219352	.00219352	.00000000	.00000000
000	.2122	-.04502883	-.06351335	.01848452	-.03753832	.00749051	-.05611114	.01108231
061	.1061	.00000000	.00589734	.00589734	.00584518	.00584518	.00000000	.00000000
122	.0000	.04502883	.01759706	.02743177	.01005942	.03496941	.05611114	.01108231
061	-.1061	.00000000	.00589735	.00589735	.00584518	.00584518	.00000000	.00000000
000	-.2122	-.04502883	-.06351334	.01848451	-.03753831	.00749052	-.05611114	.01108231
061	-.1061	.00000000	.00112405	.00112405	.00219352	.00219352	.00000000	.00000000

Table No. 30

N = 16

p) = $x^2 - y^2$

122	.0000	.04502883	.04503137	.00000254	.04502124	.00000759	.04544491	.00041608
061	.1061	.00000000	.00000570	.00000570	-.00000915	.00000915	.00000000	.00000000
000	.2122	-.04502883	-.04541993	.00039110	-.04773108	.00270225	-.04544491	.00041608
061	.1061	.00000000	-.00013213	.00013213	-.00021738	.00021738	.00000000	.00000000
122	.0000	.04502883	.04118781	.00384102	.04004087	.00498796	-.04544491	.00041608
061	-.1061	.00000000	-.00013213	.00013213	-.00021738	.00021738	.00000000	.00000000
000	-.2122	-.04502883	-.04541993	.00039110	-.04773108	.00270225	-.04544491	.00041608
061	-.1061	.00000000	.00000570	.00000570	-.00000916	.00000916	.00000000	.00000000

VALUES OF W(P)

Table No.31

(p) $= x^2 - y^2$ N = 24

rdinates of the point P X	Y	Analytic value	Computed Values Gauss-Legendre	Absolute error	Lobatto formula	Absolute error	Trapezoidal rule	Absolute error
2122	.0000	.04502883	.04502884	.00000001	.04502881	.00000002	.04504504	.00001621
1061	.1061	.00000000	.00000002	.00000002	-.00000007	.00000007	.00000000	.00000000
0000	.2122	-.04502883	-.04484032	.00018851	-.04485286	.00017597	-.04504504	.00001621
1061	.1061	.00000000	.00000142	.00000142	.00000390	.00000390	.00000000	.00000000
2122	.0000	.04502883	.04453262	.00049621	.04438382	.00064501	.04504503	.00001620
1061	-.1061	.00000000	.00000142	.00000142	.00000390	.00000390	.00000000	.00000000
0000	-.2122	-.04502883	-.04484031	.00018852	-.04485286	.00017597	-.04504504	.00001621
1061	-.1061	.00000000	.00000002	.00000002	-.00000004	.00000004	.00000000	.00000000

Table No. 32

(p) $= x^2 - y^2$ N = 32

2122	.0000	.04502883	.04502883	.00000000	.04502883	.00000000	.04502946	.00000063
1061	.1061	.00000000	.00000000	.00000000	.00000000	.00000000	.00000000	.00000000
0000	.2122	-.04502883	-.04503974	.00001091	-.04501912	.00000971	-.04502946	.00000063
1061	.1061	.00000000	.00000001	.00000001	-.00000002	.00000002	.00000000	.00000000
2122	.0000	.04502883	.04496544	.00006339	.04494650	.00008233	.04502946	.00000063
1061	-.1061	.00000000	.00000001	.00000001	-.00000002	.00000002	.00000000	.00000000
0000	-.2122	-.04502883	-.04503973	.00001090	-.04501912	.00000971	-.04502946	.00000063
1061	-.1061	.00000000	.00000000	.00000000	.00000000	.00000000	.00000000	.00000000

CHAPTER 5

DIRICHLET PROBLEM FOR RECTANGULAR CONTOUR

In the last two chapters, the possibility of applying integral equation methods for solving Dirichlet Problem in case of the circular boundary was investigated. The accuracy achieved provide enough encouragement to test them further to the case of a different boundary. One of the simplest examples where the tangent to the curve is not continuously turning is the rectangular contour.

Now so far the First Method is concerned there is no difficulty in applying it. It may however be mentioned that when q coincides p, the straight line approximation of the arc for the average value of $\log|q-p|$, gives an excellent average. On the other hand in the Second Method the kernel of one of the integral equations is $\log'|q-p|$ i.e., outward normal derivative of $\log|q-p|$ at the point q on the boundary. So even if q does not coincide with p, at the four corners of the rectangle, $\log'|q-p|$ does not exist. However this situation is already discussed in Chapter 2, but it will be worked out in some detail in this chapter, where we discuss the application of the Second Method.

As mentioned, we condider the rectangle : $x = \pm 1$, $y = \pm .5$ i.e., a rectangle of sides 2 and 1. The same two sets of boundary conditions, which were used in case of the

circular domain will be assumed here also. It is being done for two reasons. Firstly, as stated they belong to two different categories of functions, viz. even and odd and secondly, just to find out how far the two methods suggested are successful in case of the rectangular domain, as compared to that of a circle.

First Method -

The integral equation to be solved for σ is as follows.

$$\Phi(p) = -\int_{0}^{6} \sigma(q) \log|q-p| dq \qquad \ldots (85)$$

The limits have been taken from 0 to 6, because the perimeter is of length 6 units for the rectangle. In this case we solve this integral equation in a manner rightly different from that given in Chapter 3. Divide the entire contour length into N equal intervals and assume that σ is constant in each of it. Therefore

$$\Phi(p) = -\sum_{k=1}^{N} \sigma(q_k) \int_{q_{k-1/2}}^{q_{k+1/2}} \log|q-p| dq \qquad \ldots (86)$$

Replace p by q_i, $i = 1,2,\ldots,N$ where q_i is the nodal point of the interval $I_i \equiv (q_{i-1/2} , q_{i+1/2})$, thus

$$\Phi(q_i) = -\sum_{k=1}^{N} \sigma_k \int_{q_{k-1/2}}^{q_{k+1/2}} \log|q-q_i| dq \qquad \ldots (87)$$

Integrals in the above equation can be easily evaluated if $i \neq k$, otherwise we make use of the approximation (41).

Since in one of the cases, which we are considering $\Phi(q_i) = x_i$, where (x_i, y_i) are the coordinates of the point q_i on the boundary so (87) represents a set of N linear simultaneous equations in N unknowns : σ_i , $i = 1,2,\ldots,N$. As mentioned earlier the solution of this system exists and was obtained, using Gauss-elimination method for N = 12,24,36 and 48. These values are given in Table No.33,pp. 87 .

Afterwards, we replace in (86), p by P, the point where the harmonic function is to be computed. Since $\Phi(P) = V(P)$, as pointed out earlier, hence

$$V(P) = -\sum_{k=1}^{N} \sigma(q_k) \int_{q_{k-1/2}}^{q_{k+1/2}} \log|q-P| \, dq \qquad \ldots (88)$$

We simplify (88) after substituting the values of σ's and thus get the required value of V(P) . Since the domain under consideration is symmetrical about both the axis so only the positive quadrant was covered by a square grid of size .2 and V(P) was computed at each of these points apart from few points on both the axis as shown in fig.9,pp. 86 . These values along with their analytical one for above mentioned values of N are shown in Table No.34,pp. 88 . The maximum error for N=48 at any of these points is .046 % .

Similar calculations were done after replacing $\Phi(q_i)$ by $x_i^2 - y_i^2$ in (87). The values of σ's in this case are given in Table No. 35,pp. 89 , for the same values of N. Values of

V(P) at all those points were also computed and are given in Table No.36,pp. 90 . The maximum error at any point for $N = 48$ is 1.5 % .

Second Method -

The same problems were attempted by this method also to have a comparative study of the results. In this case the integral equation to be solved for $\mu(q)$ is as follows.

$$g(p) = \frac{1}{2\pi} \int_0^6 \mu(q) \log' |q-p| \, dq + \frac{1}{2} \mu(p) \qquad \ldots (89)$$

Proceeding in the same manner as described above, we find

$$g(p) = \frac{1}{2\pi} \sum_{k=1}^{N} \mu(q_k) \int_{q_{k-1/2}}^{q_{k+1/2}} \log' |q-p| \, dq + \frac{1}{2} \mu(p) \qquad \ldots (90)$$

For solving the integrals in this equation, there are two ways of replacing the integrand, as described in Chapter 2. One was found convenient in case of the circle, but we employ the second one here. Consequently

$$g(p) = \frac{1}{2\pi} \sum_{k=1}^{N} \mu(q_k) \int_{q_{k-1/2}}^{q_{k+1/2}} \frac{d\theta}{dq} \, dq + \frac{1}{2} \mu(p)$$

or

$$g(p) = \frac{1}{2\pi} \sum_{k=1}^{N} \mu_k \theta_k + \frac{1}{2} \mu(p) \qquad \ldots (91)$$

where $\mu_k = \mu(q_k)$ and $\int_{q_{k-1/2}}^{q_{k+1/2}} d\theta = \theta_k$. We can interpret θ_k as the angle traversed by the radius vector from the point p,

to the point q, as it moves from the point $q_{k-1/2}$ to $q_{k+1/2}$. It is shown in fig. 8,pp. 86 , by taking one of the nodal points as the point p. Then as usual we replace in (91), p by q_i, i = 1,2,...., N and $g(q_i)$ by x_i in the case where g(p) = x. Thus (91) can be written as

$$g(q_i) = \frac{1}{2\pi} \sum_{k \neq i} \mu_k \theta_k + \frac{1}{2\pi} \mu_i (\pi + \theta_i) .$$

It has been already proved in Chapters 2 and 4 that the coefficient matrix of this system of linear equation is non-singular. Values of μ's were obtained using Gauss elimination method for the same values of of N i.e., N = 12,24,36 and 48, and are given in Table No.37,pp. 91 . Then the value of W(P) at any point P, inside the contour may be obtained as follows.

$$W(P) = \frac{1}{2\pi} \int_0^6 \mu(q) \log' |q-P| \, dq$$

$$= \frac{1}{2\pi} \sum_{k=1}^{N} \mu(q_k) \int_{q_{k-1/2}}^{q_{k+1/2}} \log' |q-P| \, dq$$

$$= \frac{1}{2\pi} \sum_{k=1}^{N} \mu_k \theta'_k \qquad \dots (92)$$

where θ'_k is the angle subtended by the interval $I_k \equiv (q_{k-1/2}, q_{k+1/2})$ at the known point P. Different values of θ'_k for N = 12 are shown in fig.9,pp. 86 . Substituting the values of μ's in (92), we find W(P) after simplification. Computed as well as analytic values of W(P) at all those points mentioned earlier are given in Table No.38, pp. 92 . The maximum error for

$N = 48$ is .041 % . Similar computations were done for the second case where $\Phi(p) = x^2 - y^2$ and the values of σ's obtained here are given in Table No. 39,pp. 93 whereas the values of W(P) are given in Table No.40,pp. 94 . The maximum error for $N = 48$ at any point in this case is about 1.0 % .

Entire computational work was done on Russian Computer MINSK-2, at I.I.T., Bombay, where the facility for doing calculations up to 7 significant figures only is available. Autocode programmes for the case when $\Phi(p) = x^2-y^2$, by First Method, and when $g(p) = x$ by Second Method are given in Appendix III. From the errors, one can see that the Second Method in these problems is more consistant and also the maximum error with this method is less than the First Method. In the following chapters, we shall apply the foregoing theory to the investigation of some problems in Mathematical Theory of Elasticity to obtain some explicit solutions.

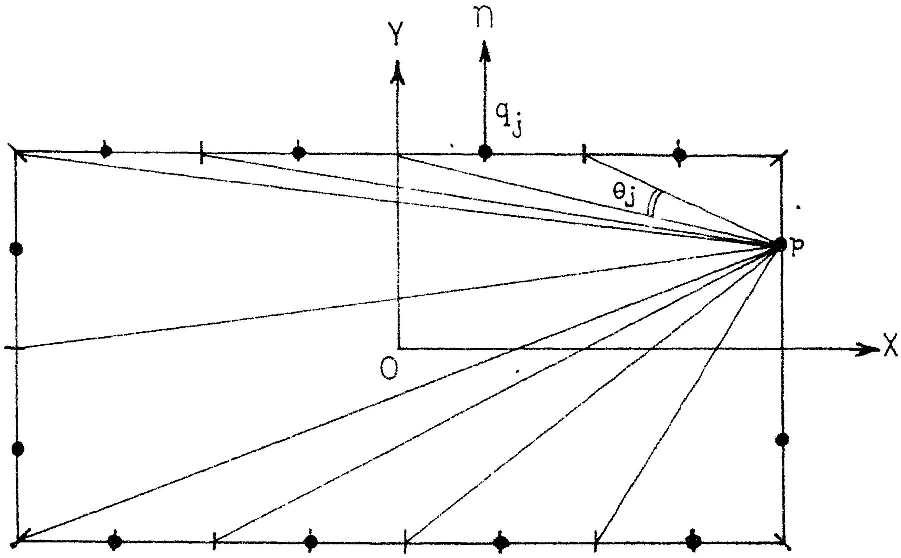

FIG. 8 – ANGLES SUBTENDED BY DIFFERENT INTERVALS OF THE BOUNDARY FOR N=12 AT ONE OF THE NODAL POINTS.

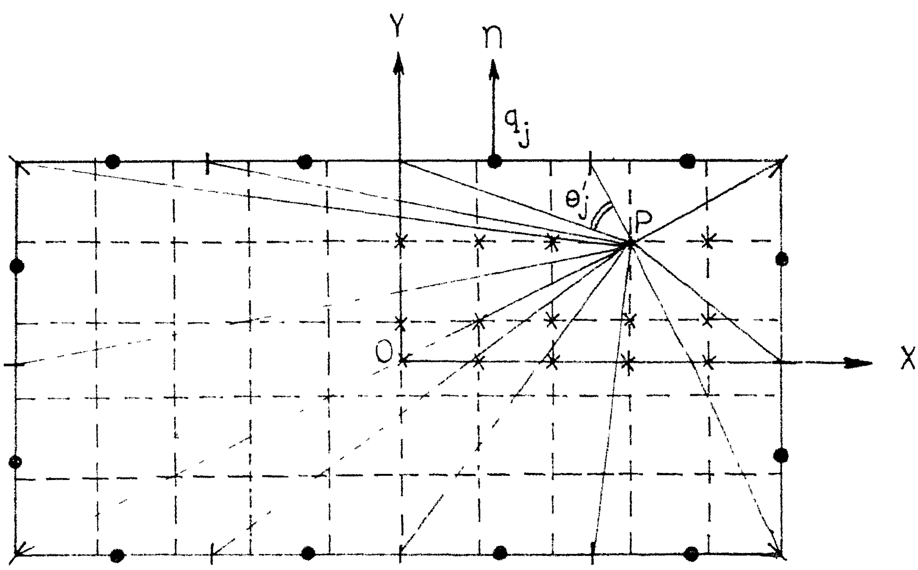

FIG. 9 – ANGLES SUBTENDED BY DIFFERENT INTERVALS OF THE BOUNDARY FOR N=12 AT ONE OF THE LATTICE POINTS OF THE NET WHERE THE SOLUTION OF THE LAPLACE EQUATION IS FOUND.

Table No. 33

COMPUTED VALUES OF σ WHEN $\Phi(p) = x$

N = 12

0.3304757	0.1926112	0.0306854	-0.0306854
-0.1926112	-0.3304757	-0.3304757	-0.1926112
-0.0306854	0.0306854	0.1926112	0.3304757

N = 24

0.2703728	0.3866806	0.2773265	0.1015816
0.0559426	0.0175114	-0.0175114	-0.0559427
-0.1015816	-0.2773265	-0.3866806	-0.2703727
-0.2703728	-0.3866806	-0.2773265	-0.1015816
-0.0559427	-0.0175114	0.0175114	0.0559426
0.1015816	0.2773265	0.3866806	0.2703728

N = 36

0.2731544	0.2833184	0.4275250	0.3327151
0.1411225	0.0972329	0.0630298	0.0358738
0.0116684	-0.0116685	-0.0358739	-0.0630298
-0.0972329	-0.1411225	-0.3327151	-0.4275249
-0.2833183	-0.2731544	-0.2731543	-0.2833184
-0.4275249	-0.3327151	-0.1411225	-0.0972329
-0.0630298	-0.0358739	-0.0116684	0.0116684
0.0358739	0.0630298	0.0972329	0.1411225
0.3327152	0.4275249	0.2837184	0.2731543

N = 48

0.2720229	0.2808361	0.2977814	0.4613241
0.3754814	0.1699174	0.1256757	0.0921602
0.0668394	0.0455772	0.0265778	0.0087395
-0.0087395	-0.0265778	-0.0455772	-0.0668394
-0.0921602	-0.1256756	-0.1699173	-0.3754813
-0.4613241	-0.2977814	-0.2808361	-0.2720229
-0.2720229	-0.2808361	-0.2977815	-0.4613241
-0.3754813	-0.1699174	-0.1256756	-0.0921601
-0.0668394	-0.0455772	-0.0265778	-0.0087395
0.0087395	0.0265778	0.0455772	0.0668394
0.0921691	0.1256757	0.1699173	0.3754813
0.4613242	0.2977814	0.2808361	0.2720229

VALUES OF V(P)

p) = x

dinates the int P Y	Analytic value	Computed value for N = 12	Absolute error	Computed value for N = 24	Absolute error	Computed value for N = 36	Absolute error	Computed value for N = 48	Absolute error
.10	.2000000	.2017052	.0017052	.2002302	.0002302	.2000654	.0000654	.2000209	.0000209
.10	.4000000	.4031836	.0031836	.4005315	.0005315	.4001550	.0001550	.4000527	.0000527
.10	.6000000	.6042670	.0042670	.6009154	.0009154	.6002730	.0002730	.6000965	.0000965
.10	.8000000	.8049704	.0049704	.8012003	.0012003	.8002944	.0002944	.8000879	.0000879
.30	.2000000	.2020482	.0020482	.2001356	.0001356	.2000405	.0000405	.2000093	.0000093
.30	.4000000	.4034251	.0034251	.4004088	.0004088	.4001141	.0001141	.4000335	.0000335
.30	.6000000	.6051185	.0051185	.6011082	.0011082	.6003017	.0003017	.6001123	.0001123
.30	.8000000	.7997985	.0002015	.8017070	.0017070	.8008194	.0008194	.8003736	.0003736
.00	.3000000	.3024430	.0024430	.3003795	.0003795	.3001097	.0001097	.3000365	.0000365
.00	.5000000	.5037303	.0037303	.5007174	.0007174	.5002116	.0002116	.5000734	.0000734
.00	.7000000	.7045231	.0045231	.7009673	.0009673	.7002797	.0002797	.7000920	.0000920
.00	.1000000	.1008534	.0008534	.1001151	.0001151	.1000326	.0000326	.1000104	.0000104
.20	.0000000	.0000000	.0000000	.0000000	.0000000	.0000000	.0000000	.0000000	.0000000
.40	.0000000	.0000000	.0000000	.0000000	.0000000	.0000000	.0000000	.0000000	.0000000
.00	.0000000	.0000000	.0000000	.0000000	.0000000	.0000000	.0000000	.0000000	.0000000

Table No. 35

COMPUTED VALUES OF σ WHEN $\Phi(\sigma) = x^2 - y^2$

N = 12

1.0919330	0.4094589	0.0048797	0.0048797
0.4094589	1.0919330	1.0919330	0.4094589
0.0048797	0.0048797	0.4094589	1.0919330

N = 24

0.9820405	1.2855030	0.7255527	0.1860558
0.0830465	0.0343775	0.0343775	0.0830466
0.1860557	0.7255527	1.2855030	0.9820406
0.9820406	1.285503	0.7255527	0.1860558
0.0830465	0.0343775	0.0343775	0.0830465
0.1860557	0.7255528	1.2855030	0.9820406

N = 36

1.0043100	1.0209670	1.4152170	0.9235413
0.3144084	0.1874686	0.1062843	0.0612819
0.0406310	0.0406310	0.0612819	0.1062843
0.1874686	0.3144084	0.9235415	1.4152160
1.0209670	1.0043100	1.0943100	1.0209670
1.4152170	0.9235413	0.3144084	0.1874686
0.1062843	0.0612819	0.0406311	0.0406311
0.0612819	0.1062843	0.1874686	0.3144083
0.9235415	1.4152170	1.0209670	1.0043100

N = 48

1.0072270	1.0257030	1.0587200	1.5201770
1.0724700	0.4118516	0.2732759	0.1788738
0.1186933	0.0792499	0.0551169	0.0436105
0.0436105	0.0551169	0.0792499	0.1186933
0.1788738	0.2732759	0.4118516	1.0724700
1.5201770	1.0587200	1.0257030	1.0072270
1.0072270	1.0257030	1.0587200	1.5201770
1.0724690	0.4118517	0.2732759	0.1788738
0.1186932	0.0792499	0.0551169	0.0436105
0.0436105	0.0551169	0.0792500	0.1186933
0.1788738	0.2732759	0.4118516	1.0724690
1.5201770	1.0587200	1.0257030	

Table No. 36

VALUES OF V(P)

p) $= x^2 - y^2$

dinates the int P	Analytic value	Computed value for	Absolute error	Computed value for	Absolute error	Computed value for	Absolute error	Computed value for	Absolute error
Y		N = 12		N = 24		N = 36		N = 48	
.10	.0300000	.0403123	.0103123	.0315811	.0015811	.0304634	.0004634	.0301605	.0001605
.10	.1500000	.1627852	.0127852	.1521549	.0021549	.1506438	.0006438	.1502282	.0002282
.10	.3500000	.3635667	.0135667	.3530054	.0030054	.3509070	.0009070	.3503209	.0003209
.10	.6299999	.6409821	.0109821	.6335260	.0035260	.6308293	.0008293	.6302171	.0002172
.30	-.0500000	-.0416322	.0083678	-.0488281	.0011719	-.0496602	.0003398	-.0498867	.0001133
.30	.0700000	.0846838	.0146838	.0717231	.0017231	.0705194	.0005194	.0701812	.0001812
.30	.2700000	.2880346	.0180346	.2736424	.0036424	.2710459	.0010459	.2704060	.0004060
.30	.5500000	.5505355	.0005355	.5551527	.0051527	.5525136	.0025136	.5511497	.0011497
.00	.0900000	.1015280	.0115280	.0918639	.0018639	.0905522	.0005522	.0901936	.0001936
.00	.2500000	.2632037	.0132037	.2525625	.0025625	.2507681	.0007681	.2502710	.0002710
.00	.4900000	.5018075	.0118075	.4929498	.0029498	.4908546	.0008546	.4902688	.0002688
.00	.0100000	.0197478	.0097478	.0114897	.0014897	.0104345	.0004345	.0101496	.0001496
.20	-.0400000	-.0316131	.0083869	-.0387374	.0012626	-.0396373	.0003627	-.0398772	.0001228
.40	-.1600000	-.1550040	.0049960	-.1593390	.0006610	-.1598330	.0001670	-.1599453	.0000547
.00	.0000000	.0094877	.0094877	.0014437	.0014437	.0004200	.0004200	.0001442	.0001442

Table No. 37

COMPUTED VALUES OF μ WHEN $g(p) = x$

N = 12

1.7885180	0.9278688	0.3042532	-0.3042531
-0.9278688	-1.7885180	-1.7885180	-0.9278688
-0.3042532	0.3042531	0.9278688	1.7885180

N = 24

1.7658940	1.6643330	1.1191940	0.7808726
0.4630147	0.1535606	-0.1535606	-0.4630147
-0.7808726	-1.1191940	-1.6643330	-1.7658930
-1.7658940	-1.6643330	-1.1191940	-0.7808726
-0.4630147	-0.1535606	0.1535606	0.4630147
0.7808726	1.1191930	1.6643330	1.7658940

N = 36

1.7601690	1.7142550	1.6095780	1.1926980
0.9520925	0.7309812	0.5170440	0.3086818
0.1026554	-0.1026554	-0.3086818	-0.5170440
-0.7309812	-0.9520925	-1.1926980	-1.6095780
-1.7142550	-1.7601690	-1.7601690	-1.7142550
-1.6095780	-1.1926980	-0.9520925	-0.7309812
-0.5170440	-0.3086818	-0.1026554	0.1026555
0.3086818	0.5170440	0.7309812	0.9520925
1.1926980	1.6095780	1.7142550	1.7601690

N = 48

1.7572730	1.7315500	1.6757400	1.5775570
1.2330880	1.0428480	0.8693944	0.7044179
0.5442565	0.3870262	0.2315750	0.0770894
-0.0770894	-0.2315750	-0.3870262	-0.5442565
-0.7044179	-0.8693944	-1.0428480	-1.2330880
-1.5775570	-1.6757400	-1.7315500	-1.7572730
-1.7572730	-1.7315500	-1.6757400	-1.5775570
-1.2330880	-1.0428480	-0.8693944	-0.7044179
-0.5442565	-0.3870262	-0.2315750	-0.0770894
0.0770894	0.2315750	0.3870262	0.5442565
0.7044179	0.8693944	1.0428480	1.2330880
1.5775570	1.6757400	1.7315500	1.7572730

Table No. 38

VALUES OF W(P)

:p) = x

dinates f the oint P		Analytic value	Computed value for	Absolute error	Computed value for	Absolute error	Computed value for	Absolute error	Computed value for	Absolute error
	Y		N = 12		N = 24		N = 36		N = 48	
0	.10	.2000000	.2001989	.0001989	.1999252	.0000748	.1999644	.0000356	.1999791	.0000209
0	.10	.4000000	.3987845	.0012155	.3997860	.0002140	.3998960	.0001040	.3999390	.0000610
0	.10	.6000000	.5990817	.0009183	.5994801	.0005199	.5997444	.0002556	.5998501	.0001499
0	.10	.8000000	.7948987	.0051013	.7989135	.0010865	.7994470	.0005530	.7996733	.0003267
0	.30	.2000000	.2042370	.0042370	.1996848	.0003152	.2000107	.0000107	.1999959	.0000041
0	.30	.4000000	.3922933	.0077067	.3997494	.0002506	.3999835	.0000165	.3999856	.0000144
0	.30	.6000000	.6078152	.0078152	.5999812	.0000188	.5998924	.0001076	.5999430	.0000570
0	.30	.8000000	.7962643	.0037357	.7994848	.0005152	.7996189	.0003811	.7997794	.0002206
0	.00	.3000000	.2993992	.0006008	.2998570	.0001430	.2999299	.0000701	.2999589	.0000411
0	.00	.5000000	.4989167	.0010833	.4996327	.0003673	.4998203	.0001797	.4998946	.0001054
0	.00	.7000000	.6969975	.0030025	.6991619	.0008381	.6995959	.0004041	.6997635	.0002365
0	.00	.1000000	.1002416	.0002416	.0999636	.0000364	.0999821	.0000179	.0999895	.0000105
0	.20	.0000000	.0000000	.0000000	.0000000	.0000000	.0000000	.0000000	.0000000	.0000000
0	.40	.0000000	.0000000	.0000000	.0000000	.0000000	.0000000	.0000000	.0000000	.0000000
0	.00	.0000000	.0000000	.0000000	.0000000	.0000000	.0000000	.0000000	.0000000	.0000000

Table No. 39

COMPUTED VALUES OF μ WHEN $g(p) = x^2 - y^2$

N = 12

2.0010850	-0.2770791	-1.0430350	-1.0430350
-0.2770791	2.0010850	2.0010850	-0.2770791
-1.0430350	-1.0430350	-0.2770791	2.0010850

N = 24

2.0163270	1.6603820	0.1581172	-0.4812622
-0.8765929	-1.0675480	-1.0675480	-0.8765930
-0.4812622	0.1581172	1.6603820	2.0163270
2.0163270	1.6603820	0.1581172	-0.4812622
-0.8765930	-1.0675480	-1.0675480	-0.8765930
-0.4812622	0.1581172	1.6603820	2.0163270

N = 36

2.0151260	1.8550440	1.5027770	0.3413849
-0.1721810	-0.5454077	-0.8111952	-0.9832994
-1.0679920	-1.0679920	-0.9832994	-0.8111952
-0.5454077	-0.1721810	0.3413849	1.5027770
1.8550440	2.0151260	2.0151260	1.8550440
1.5027770	0.3413849	-0.1721810	-0.5454077
-0.8111952	-0.9832994	-1.0679920	-1.0679920
-0.9832094	-0.8111952	-0.5454077	-0.1721810
0.3413850	1.5027770	1.8550440	2.0151260

N = 48

2.0126740	1.9227980	1.7322070	1.4029050
0.4455966	0.0144707	-0.3169865	-0.5765871
-0.7765859	-0.9230350	-1.0192180	-1.0669150
-1.0669150	-1.0192180	-0.9230350	-0.7765859
-0.5765871	-0.3169865	0.0144797	0.4455966
1.4029050	1.7322070	1.9227980	2.0126740
2.0126740	1.9227980	1.7322070	1.4099050
0.4455966	0.0144707	-0.3169865	-0.5765871
-0.7765859	-0.9230350	-1.0192180	-1.0669150
-1.0669150	-1.0192180	-0.9230350	-0.7765859
-0.5765871	-0.3169865	0.0144707	0.4455966
1.4029050	1.7322070	1.9227980	2.0126740

Table No. 40

VALUES OF W(P)

:p) $= x^2 - y^2$

rdinates f the oint P		Analytic value	Computed value for N = 12	Absolute error	Computed value for N = 24	Absolute error	Computed value for N = 36	Absolute error	Computed value for N = 48	Absolute error
	Y									
0	.10	.0300000	.0408493	.0108493	.0328310	.0028310	.0312621	.0012621	.0307111	.0007111
0	.10	.1500000	.1582922	.0082922	.1521071	.0021071	.1509336	.0009336	.1505243	.0005243
0	.10	.3500000	.3533512	.0033512	.3504851	.0004851	.3501845	.0001845	.3500960	.0000960
30	.10	.6299999	.6142700	.0157299	.6275289	.0024710	.6287405	.0012594	.6292569	.0007430
20	.30	-.0500000	-.0391421	.0108579	-.0467821	.0032179	-.0485106	.0014894	-.0491668	.0008332
40	.30	.0700000	.0760629	.0060629	.0726131	.0026131	.0713283	.0013283	.0707506	.0007506
50	.30	.2700000	.2929424	.0229424	.2721476	.0021476	.2708980	.0008980	.2705220	.0005220
30	.30	.5500000	.5465955	.0034045	.5495888	.0004112	.5496353	.0003647	.5497867	.0002133
30	.00	.0900000	.0994060	.0094060	.0924680	.0024680	.0910964	.0010964	.0906169	.0006169
50	.00	.2500000	.2558478	.0058478	.2512962	.0012962	.2505590	.0005590	.2503101	.0003101
70	.00	.4900000	.4835933	.0064067	.4888882	.0011118	.4894629	.0005371	.4896826	.0003174
10	.00	.0100000	.0219212	.0119212	.0129292	.0029292	.0113066	.0013066	.0107363	.0007363
00	.20	-.0400000	-.0252918	.0147082	-.0367999	.0032001	-.0385803	.0014197	-.0391997	.0008003
00	.40	-.1600000	-.1318805	.0281195	-.1549289	.0050711	-.1581259	.0018741	-.1590289	.0009711
00	.00	.0000000	.0123823	.0123823	.0029821	.0029821	.0013302	.0013302	.0007497	.0007497

CHAPTER 6

TORSION PROBLEM FOR PRISMS OF RECTANGULAR AND EQUILATERAL TRIANGULAR CROSS-SECTIONS

The torsion problem of an elastic cylinder is one of the classical problems of the theory of elasticity. The details of the problem and its equivalent mathematical formulations are given in all the standard books on Theory of Elasticity [50]. The main results of the theory are discussed below in brief.

Consider a cylinder of homogeneous isotropic elastic material of rigidity μ. One end of the cylinder is fixed and the other end is being twisted by a couple of moment M, about an axis parallel to the generators of the cylinder. The origin of the coordinate system is at the fixed end. The z-axis is parallel to the axis of the cylinder and x and y axis are any two mutually perpendicular axis in the fixed plane. To solve the problem the following displacements are assumed,

$$u = \alpha yz \ , \quad v = \alpha xz \ , \quad w = \alpha \overset{*}{\Phi}(x,y) . \qquad \ldots (93)$$

Here α is the angle of twist per unit length and is a constant. The function $\overset{*}{\Phi}(x,y)$ is the unknown function to be determined and is called the warping or torsion function. The torsion problem consists of finding $\overset{*}{\Phi}(x,y)$ or an equivalent function, subject to suitable boundary conditions.

Displacements in (93) give the following two non-zero

strain components

$$e_{zx} = \alpha\left(\frac{\partial\Phi^*}{\partial x} - y\right) \; ; \quad e_{yz} = \alpha\left(\frac{\partial\Phi^*}{\partial y} + x\right) \qquad \ldots (94)$$

whence the stresses are

$$\tau_{zx} = \mu\alpha\left(\frac{\partial\Phi^*}{\partial x} - y\right) \; ; \quad \tau_{yz} = \mu\alpha\left(\frac{\partial\Phi^*}{\partial y} + x\right) \qquad \ldots (95)$$

The equations of equilibrium [50], when there are no body forces reduce to

$$\nabla^2\Phi^* \equiv \frac{\partial^2\Phi^*}{\partial x^2} + \frac{\partial^2\Phi^*}{\partial y^2} = 0 \qquad \ldots (96)$$

It can be proved that since the surface of the cylinder is free from traction, therefore

$$\frac{\partial\Phi^*}{\partial n} = y\cos(x,n) - x\cos(y,n) \quad \text{on } C, \qquad \ldots (97)$$

where C is the boundary of the cross-section of the cylinder. Thus by (96) and (97) the torsion problem is formulated as a Neumann Problem. It can be also formulated as a Dirichlet Problem. Let Ψ be the harmonic conjugate of Φ^*; thus $\Phi^* + i\Psi$ is an analytic function of $x + iy$. Whence it is well known that

$$\nabla^2\Psi = 0$$

and

$$\frac{\partial\Phi^*}{\partial n} = \frac{\partial\Psi}{\partial s} \qquad \ldots (98)$$

Noting that $\cos(x,n) = \frac{dy}{ds}$ and $\cos(y,n) = \frac{-dx}{ds}$, it may be seen that $\Psi = 1/2\,(x^2+y^2) +$ const. on the boundary. For simply -

connected regions this constant may be taken as zero.

Thus the torsion problem can be formulated in two ways either as a Neumann Problem or as a Dirichlet Problem. In this chapter, numerical solution of the torsion problems for prisms of rectangular and triangular cross-sections, by the methods given in Chapter 2, are obtained. These are compared with the analytical solutions. The results seem to be very encouraging.

Rectangular Cross-Section -

The sides of the rectangle are taken to be 2 and 1 ; the origin of the coordinate system at the centre and axis of x and y parallel to the sides of the rectangle as shown in fig. 8, pp. 86 . The analytical solution of this problem is available in the series form [50], which is as follows.

$$\psi(x,y) = \frac{1}{4} + \left(\frac{y^2-x^2}{2}\right) - \frac{8}{\pi^3}\sum_{j=o}^{\infty} \frac{(-1)^j}{(2j+1)^3}\,\frac{\cosh(2j-1)\pi y}{\cosh(2j-1)\pi}\cos(2j-1)\pi x \quad \ldots (99)$$

We now give very briefly the computational steps of the two methods seperately.

First Method -

The values of σ_k are to be obtained from the following system of linear equations

$$\frac{1}{2}(x_i^2 + y_i^2) = -\sum_{k=1}^{N} \sigma_k \int_{q_{k-1/2}}^{q_{k+1/2}} \log\left|q-q_i\right| dq \quad \ldots (100)$$

where (x_i, y_i) are the coordinates of the nodal points q_i, $i = 1,2,\ldots,N$ on the boundary. Approximation (41) was used for evaluating the integrals in (100) and then the system was solved by Gauss elimination method. The values of σ_k as shown in Table No.41,pp. 104 ,were substituted in (88) , which in the new notation is

$$\Psi(P) = -\sum_{k=1}^{N} \sigma_k \int_{q_{k-1/2}}^{q_{k+1/2}} \log|q-P|\,dq \qquad \ldots (101)$$

The positive quadrant was covered by square net of side .1 and was computed at each of these grid points (fig.10,pp. 102). The number of nodal points that were taken on the boundary was N, where N = 12,24,36 and 48 successively. The rate of convergence of Ψ with this method is given in Table No.42,pp. 105 . The maximum error at any point for N = 48 is .62 % .

Second Method -

The same results were obtained by the Second Method also. In this case the values of μ_k are the solutions of the following system of linear equations.

$$\pi(x_i^2 + y_i^2) = \sum_{k\neq i}^{N} \mu_k \theta_k + (\pi+\theta_i)\mu_i, \quad i=1,2,\ldots,N. \qquad \ldots (102)$$

All the symbols involved in these equations have already been explained earlier. Computed values of μ's for above mentioned values of N are shown in Table No.43,pp. 108 . Values of $\Psi(P)$ are obtained from (92) which in the present notation is

$$\Psi(P) = \frac{1}{2\pi} \sum_{k=1}^{N} \mu_k \, \theta_k' \qquad \ldots (103)$$

The values of Ψ were computed again at all those points where they were computed by First Method, and are given in Table No.44, pp. 109 ,for different values of N described earlier. This table also indicates the rate of convergence of Ψ by this method. The maximum error for N = 48 at any of these points is .10 % . Stress function Ψ which is defined as follows.

$$\Psi = \psi - \frac{1}{2} r^2,$$

where r is the distance of a boundary point from the origin in the plane of the cross-section, was computed at all the lattice points of the net in the positive quadrant for N = 48. Curves given by Ψ = const., known as lines of shearing stress are drawn in fig.10, pp. 102 .

Equilateral Triangular Cross- Section -

This problem was also done by both the methods. Analytical solution of this problem is available [50]. If the cross-section is an equilateral triangle of altitude 3a, ψ is given by

$$\psi(x,y) = -\frac{1}{6a}(x^3 - 3xy^2) + \frac{2a^2}{3} \qquad \ldots (104)$$

For numerical computation we took $a = 1/\sqrt{3}$, so that each side is of length 2.

First Method -

In this case also the contour length was divided into 12,24,36 and 48 equal intervals successively. The values of σ_k were obtained from (100), after substituting appropriate values of the nodal points and end points of the intervals, and then using approximation (41). These values appear in Table No.45,pp. 112. One half of the cross-section about the axis of symmetry was covered by a triangular net and Ψ was computed at all the lattice points of the net (fig.12,pp. 103)for values of N mentioned above. These values are given in Table No. 46, pp. 113 along with the error at each point for different values of N. The maximum error at any point for N = 48 is .65 %

Second Method -

With the coordinates of the end points and nodal points already known, the values of μ_k were computed from (102) as shown in Table No.47,pp. 116 and then substituted in (103) to find Ψ (P) at all grid points described above. Values of θ_j in (102) and of θ_j' in (103) for N = 12 are shown in Fig.Nos. 11 and 12, pps. 103 . The error in the value of Ψ at any point of the net for values of N mentioned above can be found from Table No. 48, pp. 117 . The maximum error at any point for N = 48 is .43 % .

The maximum percentage error in these two problems show that the results are almost identical by First and Second Methods, but perhaps the Second Method is slightly better.

Complete calculation work for these problems was done in a single autocode programme, run on the Russian Computer MINSK-2. Autocode programmes for the case of the triangular cross-section by both the methods seperately are given in Appendix IV.

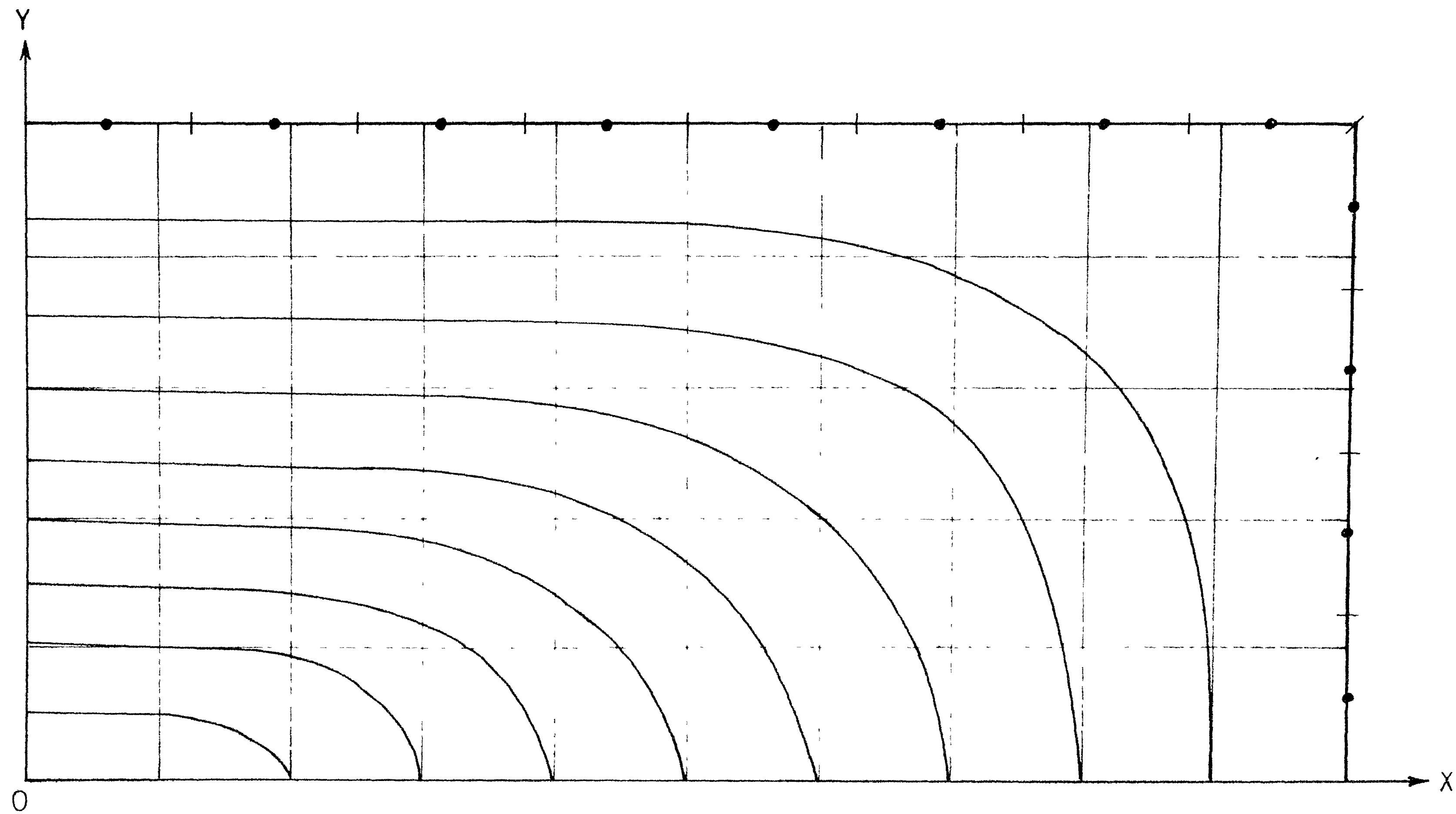

FIG. 10 - LINES OF SHEARING STRESS IN THE POSITIVE QUADRANT OF THE RECTANGULAR CROSS-SECTION OF SIDES 2 X 1.

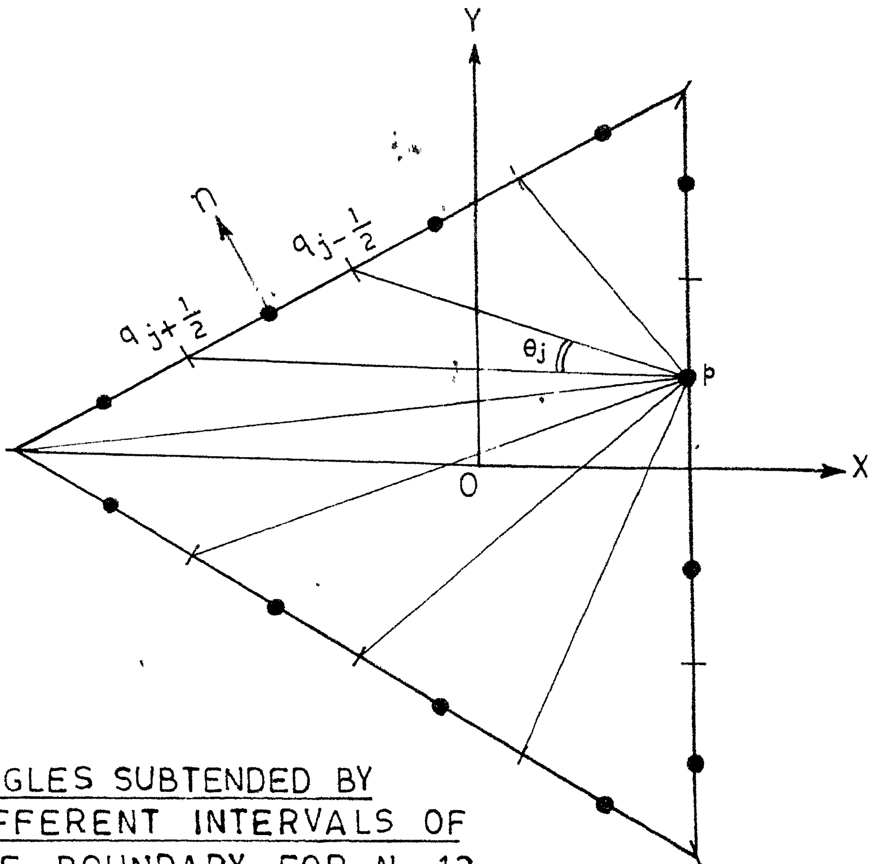

FIG.11- ANGLES SUBTENDED BY DIFFERENT INTERVALS OF THE BOUNDARY FOR N=12 AT ONE OF THE NODAL POINTS.

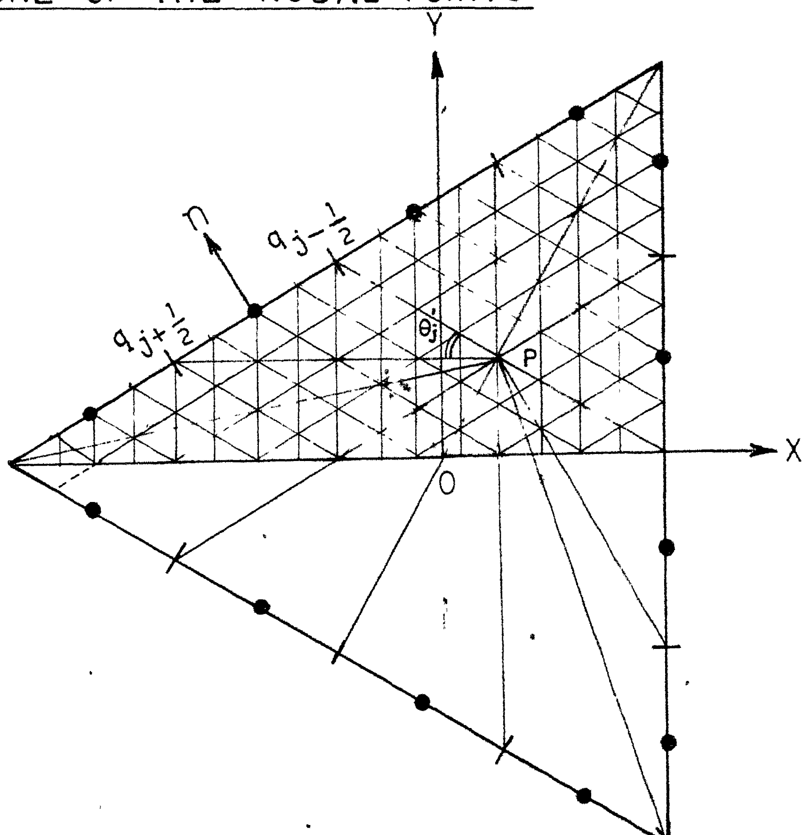

FIG. 12- ANGLES SUBTENDED BY DIFFERENT INTERVALS OF THE BOUNDARY FOR N=12 AT ONE OF THE LATTICE

Table No. 41

TORSION PROBLEM FOR RECTANGLE

Computed Values of σ

N = 12

.6501155	.5298066	.1539484	.1539484	.5298066	.6501155
.6501154	.5298066	.1539484	.1539484	.5298066	.6501155

N = 24

.4940400	.9250590	.8235747	.3189098	.2296635	.1882329
.1882329	.2296635	.3189098	.8235747	.9250590	.4940400
.4940400	.9250591	.8235746	.3189099	.2296635	.1882329
.1882329	.2296635	.3189098	.8235747	.9250591	.4940400

N = 36

.5124586	.5760000	1.0923870	1.0044400	.4363835	.3228613
.2513018	.2126460	.1952068	.1952068	.2126460	.2513018
.3228613	.4363834	1.0044400	1.0923870	.5760000	.5124586
.5124586	.5760000	1.0923870	1.0044400	.4363835	.3228613
.2513019	.2126460	.1952068	.1952069	.2126459	.2513019
.3228613	.4363835	1.0044400	1.0923870	.5760001	.5124585

N = 48

.5136566	.5580346	.6482793	1.2194240	1.1398300	.5256419
.4004918	.3157195	.2627165	.2286179	.2080747	.1983774
.1983774	.2080747	.2286179	.2627165	.3157195	.4004917
.5256420	1.1398300	1.2194240	.6482793	.5580346	.5136567
.5136566	.5580347	.6482793	1.2194240	1.1398300	.5256420
.4004918	.3157195	.2627165	.2286179	.2080747	.1983774
.1983774	.2080748	.2286179	.2627165	.3157195	.4004918
.5256420	1.1398290	1.2194240	.6482794	.5580345	.5136567

CONJUGATE TORSION FUNCTION (Ψ) FOR RECTANGLE OF SIDES 1 x 2

BY FIRST METHOD

dinates of points P		Analytic value of Ψ	Computed value of Ψ for N = 12	Absolute error	Computed value of Ψ for N = 24	Absolute error	Computed value of Ψ for N = 36	Absolute error	Computed value of Ψ for N = 48	Absolute error
	Y									
00	0.00	.2277439	.2371187	.0093748	.2259532	.0017907	.2263421	.0014018	.2276212	.0001227
10	0.00	.2316372	.2413994	.0097622	.2330692	.0014320	.2320098	.0003726	.2317344	.0000972
20	0.00	.2432076	.2540019	.0107943	.2448141	.0016065	.2436399	.0004323	.2433312	.0001236
30	0.00	.2621162	.2742984	.0121822	.2640197	.0019035	.2626502	.0005340	.2622848	.0001686
40	0.00	.2877636	.3014397	.0136761	.2900862	.0023226	.2884411	.0006775	.2879960	.0002324
50	0.00	.3192363	.3344133	.0151770	.3220745	.0028382	.3200904	.0008541	.3195471	.0003108
60	0.00	.3552311	.3720591	.0168280	.3585985	.0033674	.3562645	.0010334	.3556205	.0003894
70	0.00	.3939638	.4131992	.0192354	.3976801	.0037163	.3951059	.0011421	.3943968	.0004330
80	0.00	.4330787	.4566945	.0236158	.4364311	.0033524	.4341227	.0010440	.4334604	.0003817
90	0.00	.4696017	.5048228	.0352211	.4691107	.0004910	.4696853	.0000836	.4696508	.0000491
00	0.10	.2238327	.2327951	.0089624	.2251496	.0013169	.2241660	.0003333	.2239127	.0000800
10	0.10	.2277798	.2372046	.0094248	.2291539	.0013741	.2281324	.0003526	.2278683	.0000885
20	0.10	.2395170	.2501390	.0106220	.2410618	.0015448	.2399283	.0004113	.2396313	.0001143
30	0.10	.2587209	.2708636	.0121427	.2605616	.0018407	.2592335	.0005126	.2588799	.0001590
40	0.10	.2848189	.2985081	.0136892	.2870884	.0022695	.2854780	.0006591	.2850431	.0002242
50	0.10	.3169351	.3320765	.0151414	.3197580	.0028229	.3177844	.0008493	.3172442	.0003091

Table No. 42 (CONTD.)

dinates of points P		Analytic value of ψ	Computed value of ψ for N = 12	Absolute error	Computed value of ψ for N = 24	Absolute error	Computed value of ψ for N = 36	Absolute error	Computed value of ψ for N = 48	Absolute error
	Y									
60	0.10	.3538140	.3703209	.0165069	.3572549	.0034409	.3548775	.0010635	.3542184	.0004044
70	0.10	.3937197	.4120104	.0182907	.3976951	.0039754	.3949579	.0012382	.3941991	.0004794
80	0.10	.4343257	.4562348	.0219091	.4386818	.0043561	.4355627	.0012370	.4347893	.0004636
90	0.10	.4726335	.4959745	.0233410	.4787889	.0061554	.4739134	.0012799	.4728450	.0002115
00	0.20	.2119928	.2195479	.0075551	.2131354	.0011426	.2122686	.0002758	.2120476	.0000548
10	0.20	.2160966	.2244018	.0083052	.2173033	.0012067	.2163901	.0002935	.2161590	.0000624
20	0.20	.2283187	.2384771	.0101584	.2296754	.0013567	.2286666	.0003479	.2284047	.0000860
30	0.20	.2483819	.2604950	.0121131	.2500243	.0016424	.2488268	.0004449	.2485105	.0001286
40	0.20	.2757930	.2895937	.0138007	.2778789	.0020859	.2763880	.0005950	.2759882	.0001952
50	0.20	.3097917	.3250556	.0152639	.3125126	.0027209	.3106051	.0008134	.3100854	.0002937
60	0.20	.3492713	.3652495	.0159782	.3528728	.0036015	.3503904	.0011191	.3497040	.0004327
70	0.20	.3926603	.4081305	.0154702	.3971388	.0044785	.3941607	.0015004	.3932685	.0006082
80	0.20	.4377572	.4544796	.0167224	.4430302	.0052730	.4395739	.0018167	.4385033	.0007461
90	0.20	.4815308	.5209285	.0393977	.4872185	.0056877	.4830005	.0014697	.4822146	.0006838
00	0.30	.1919159	.1964706	.0045547	.1927064	.0007905	.1920947	.0001788	.1919303	.0000144
10	0.30	.1962643	.2021617	.0058974	.1972641	.0009998	.1964664	.0002021	.1962847	.0000204
20	0.30	.2092439	.2190194	.0097755	.2102565	.0010126	.2094855	.0002416	.2092834	.0000395

Table No. 42 (CONTD.)

dinates of points P		Analytic value of ψ	Computed value of ψ for N = 12	Absolute error	Computed value of ψ for N = 24	Absolute error	Computed value of ψ for N = 36	Absolute error	Computed value of ψ for N = 48	Absolute error
	Y									
.30	0.30	.2306514	.2431142	.0124628	.2319323	.0012809	.2309737	.0003223	.2307262	.0000748
.40	0.30	.2601228	.2740596	.0139368	.2618734	.0017506	.2605921	.0004693	.2602574	.0001346
.50	0.30	.2970914	.3132792	.0161878	.2993577	.0022663	.2977725	.0006811	.2973275	.0002361
.60	0.30	.3407153	.3575471	.0168318	.3445391	.0038238	.3417980	.0010827	.3411306	.0004513
.70	0.30	.3897533	.4032813	.0135275	.3948954	.0051416	.3915414	.0017876	.3904963	.0007425
.80	0.30	.4423440	.4385694	.0037746	.4482524	.0059084	.4451006	.0027566	.4436253	.0012813
.90	0.30	.4955949	.4813135	.0142814	.5024008	.0068059	.4993312	.0037363	.4974962	.0019013
.00	0.40	.1631211	.1628683	.0002528	.1626203	.0005008	.1629554	.0001657	.1630324	.0000887
.10	0.40	.1677782	.1673035	.0004747	.1693738	.0015956	.1680437	.0002655	.1677258	.0000524
.20	0.40	.1817150	.1933213	.0116063	.1817661	.0000511	.1817482	.0000332	.1817343	.0000193
.30	0.40	.2048238	.2195040	.0146802	.2052333	.0004095	.2048565	.0000327	.2048732	.0000494
.40	0.40	.2369109	.2482704	.0113595	.2389800	.0020691	.2374339	.0005230	.2369337	.0000228
.50	0.40	.2776718	.2984130	.0207412	.2776324	.0000394	.2777616	.0000898	.2777058	.0000340
.60	0.40	.3266463	.3442969	.0176506	.3317259	.0050796	.3276249	.0009786	.3268624	.0002161
.70	0.40	.3831321	.4145818	.0314497	.3889922	.0058601	.3847195	.0015874	.3838087	.0006766
.80	0.40	.4459947	.4249696	.0210251	.4535382	.0075435	.4498268	.0038321	.4479176	.0019229
.90	0.40	.5131588	.3683782	.1447806	.5098218	.0033370	.5176371	.0044783	.5163727	.0032139

Table No. 43

TORSION PROBLEM FOR RECTANGLE

Computed Values of μ

N = 12

.7909926	.3094295	-.1208199	-.1208199	.3094295	.7909926
.7909926	.3094295	-.1208199	-.1208199	.3094295	.7909926

N = 24

.7236394	.7913501	.4921308	.1684519	-.0445960	-.1496442
-.1496442	-.0445960	.1684519	.4921308	.7913501	.7236394
.7236394	.7913502	.4921308	.1684519	-.0445960	-.1496442
-.1496442	-.0445960	.1684519	.4921308	.7913502	.7236394

N = 36

.7122499	.7423412	.7875809	.5597822	.3202039	.1287582
-.0134327	-.1072476	-.1538156	-.1538156	-.1072476	-.0134327
.1287582	.3202039	.5597822	.7875810	.7423412	.7122499
.7122499	.7423412	.7875809	.5597822	.3202039	.1287582
-.0134327	-.1072476	-.1538156	-.1538156	-.1072476	-.0134327
.1287582	.3202039	.5597822	.7875809	.7423412	.7122498

N = 48

.7080194	.7252595	.7548406	.7831425	.5952637	.4066801
.2444249	.1098973	.0031539	-.0762458	-.1288253	-.1550023
-.1550023	-.1288253	-.0762458	.0031539	.1098973	.2444249
.4066801	.5952637	.7831425	.7548406	.7252595	.7080194
.7080194	.7252596	.7548405	.7831425	.5952637	.4066801
.2444249	.1098973	.0031539	-.0762458	-.1288253	-.1550023
-.1550023	-.1288253	-.0762458	.0031539	.1098973	.2444249
.4066801	.5952637	.7831425	.7548406	.7252596	.7080194

CONJUGATE TORSION FUNCTION (ψ) FOR RECTANGLE OF SIDES 1 x 2

BY SECOND METHOD

inates of points P		Analytic value of ψ	Computed value of ψ for N = 12	Absolute error	Computed value of ψ for N = 24	Absolute error	Computed value of ψ for N = 36	Absolute error	Computed value of ψ for N = 48	Absolute error
	Y									
0	0.00	.2277439	.2364463	.0087024	.2298448	.0021009	.2286680	.0009241	.2282603	.0005164
0	0.00	.2316372	.2401708	.0085336	.2337329	.0020957	.2325587	.0009215	.2321519	.0005147
0	0.00	.2432076	.2513793	.008171	.2452884	.0020808	.2441207	.0009131	.2437170	.0005094
0	0.00	.2621162	.2700535	.0079373	.2641694	.0020532	.2630142	.0008980	.2626162	.0005000
0	0.00	.2877636	.2957941	.0080305	.2897724	.0020088	.2886380	.0008744	.2882488	.0004852
0	0.00	.3192363	.3275132	.0082769	.3211819	.0019456	.3200765	.0008402	.3197003	.0004640
0	0.00	.3552311	.3635474	.0083163	.3570915	.0018604	.3560258	.0007947	.3556671	.0004360
0	0.00	.3939638	.4022742	.0083104	.3957290	.0017652	.3947042	.0007404	.3943668	.0004030
0	0.00	.4330787	.4425896	.0095109	.4348117	.0017330	.4337692	.0006905	.4334506	.0003719
0	0.00	.4696017	.4836580	.0140563	.4717634	.0021617	.4703509	.0007492	.4699763	.0003746
0	0.10	.2238327	.2327313	.0088986	.2259392	.0021065	.2247593	.0009266	.2243507	.0005180
0	0.10	.2277798	.2364275	.0086477	.2298799	.0021001	.2287041	.0009243	.2282963	.0005165
0	0.10	.2395170	.2476153	.0080983	.2416039	.0020869	.2404335	.0009165	.2400286	.0005116
0	0.10	.2587209	.2664563	.0077354	.2607832	.0020623	.2596233	.0009024	.2592236	.0005027
0	0.10	.2848189	.2927471	.0079282	.2868363	.0020174	.2856985	.0008796	.2853073	.0004884
0	0.10	.3169351	.3253610	.0084259	.3188930	.0019579	.3177813	.0008462	.3174027	.0004676

CONTD.

Table No. 44 (CONTD.)

Co-ordinates of points P		Analytic value of Ψ	Computed value of Ψ for N = 12	Absolute error	Computed value of Ψ for N = 24	Absolute error	Computed value of Ψ for N = 36	Absolute error	Computed value of Ψ for N = 48	Absolute error
	Y									
0	0.10	.3538140	.3622577	.0084437	.3556826	.0018686	.3546138	.0007998	.3542529	.0004389
0	0.10	.3937197	.4015092	.0077895	.3954701	.0017504	.3944605	.0007408	.3941226	.0004029
0	0.10	.4343256	.4421736	.0078480	.4359217	.0015961	.4349977	.0006721	.4346901	.0003645
0	0.10	.4726335	.4838274	.0111939	.4739682	.0013347	.4731450	.0005115	.4729610	.0003275
0	0.20	.2119928	.2216640	.0096712	.2141230	.0021302	.2129269	.0009341	.2125154	.0005226
0	0.20	.2160966	.2252115	.0091149	.2182042	.0021076	.2170286	.0009320	.2166180	.0005214
0	0.20	.2283187	.2361446	.0078259	.2304209	.0021022	.2292451	.0009264	.2288364	.0005177
0	0.20	.2483819	.2552410	.0068591	.2504804	.0020985	.2492968	.0009149	.2488924	.0005105
0	0.20	.2757930	.2831830	.0073900	.2778248	.0020318	.2766888	.0008958	.2762914	.0004984
0	0.20	.3097917	.3189487	.0091570	.3117978	.0020061	.3106571	.0008654	.3102708	.0004791
0	0.20	.3492713	.3588289	.0095576	.3511787	.0019074	.3500893	.0008180	.3497211	.0004498
0	0.20	.3926603	.3994405	.0067802	.3944116	.0017513	.3934105	.0007502	.3930671	.0004068
0	0.20	.4377572	.4406343	.0028771	.4393374	.0015802	.4384172	.0006600	.4381050	.0003478
0	0.20	.4815308	.4838584	.0023276	.4822545	.0007237	.4822391	.0007083	.4817845	.0002537
0	0.30	.1919159	.2036010	.0116851	.1941576	.0022417	.1928659	.0009500	.1924457	.0005298
0	0.30	.1962643	.2067015	.0104372	.1983415	.0020772	.1972025	.0009382	.1967932	.0005289
0	0.30	.2092439	.2165310	.0072871	.2113267	.0021828	.2101913	.0009474	.2097706	.0005267

Table No. 44 (CONTD.)

dinates of points P	Y	Analytic value of ψ	Computed value of ψ for N = 12	Absolute error	Computed value of ψ for N = 24	Absolute error	Computed value of ψ for N = 36	Absolute error	Computed value of ψ for N = 48	Absolute error
30	0.30	.2306514	.2349119	.0042605	.2329310	.0022796	.2315817	.0009303	.2311745	.0005231
40	0.30	.2601228	.2651671	.0050443	.2620352	.0019124	.2610470	.0009242	.2606387	.0005159
50	0.30	.2970914	.3084776	.0113862	.2992634	.0021720	.2979973	.0009059	.2975924	.0005010
60	0.30	.3407153	.3553995	.0146842	.3427669	.0020516	.3415611	.0008458	.3411890	.0004737
70	0.30	.3897538	.3970805	.0073267	.3913457	.0015919	.3905574	.0008036	.3901806	.0004268
80	0.30	.4423440	.4367437	.0056003	.4444397	.0020957	.4429775	.0006335	.4426887	.0003447
.90	0.30	.4955949	.4814302	.0141647	.4976429	.0020480	.4958620	.0002671	.4958195	.0002246
.00	0.40	.1631211	.1794984	.0163773	.1660577	.0029366	.1642081	.0010870	.1636844	.0005633
.10	0.40	.1677782	.1816674	.0138892	.1696326	.0018544	.1685725	.0007943	.1683030	.0005248
.20	0.40	.1817150	.1886772	.0069622	.1830461	.0013311	.1829095	.0011945	.1822068	.0004918
.30	0.40	.2048238	.2027342	.0020896	.2087885	.0039647	.2055301	.0007063	.2053802	.0005564
.40	0.40	.2369109	.2315409	.0053700	.2372626	.0003517	.2380019	.0010910	.2375420	.0006311
.50	0.40	.2776718	.2944857	.0168139	.2806563	.0029845	.2787627	.0010909	.2782328	.0005610
60	0.40	.3266463	.3599954	.0333491	.3299526	.0033063	.3271013	.0004550	.3270450	.0003987
70	0.40	.3831321	.3970611	.0139290	.3815790	.0015531	.3847575	.0016254	.3835003	.0003682
80	0.40	.4459947	.4275453	.0184494	.4522448	.0062501	.4459785	.0000162	.4464822	.0004875
90	0.40	.5131588	.4677765	.0453823	.5099822	.0031766	.5141775	.0010187	.5135850	.0004262

Table No. 45

TORSION PROBLEM FOR EQUILATERAL TRIANGLE

Computed Values of σ

N = 12

.1082295	.5059800	.5059800	.1082295	.1082295	.5059800
.5059800	.1082295	.1082295	.5059800	.5059800	.1082295

N = 24

.1368034	.1862622	.2816941	.8004660	.8004659	.2816941
.1862622	.1368034	.1368034	.1862622	.2816941	.8004660
.8004660	.2816940	.1862622	.1368034	.1368034	.1862623
.2816941	.8004659	.8004660	.2816941	.1862622	.1368034

N = 36

.1423724	.1635063	.2085517	.2871443	.3983969	.9897470
.9897468	.3983969	.2871443	.2085517	.1635063	.1423725
.1423724	.1635063	.2085517	.2871443	.3983969	.9897469
.9897468	.3983969	.2871443	.2085517	.1635063	.1423724
.1423725	.1635063	.2085517	.2871444	.3983968	.9897468
.9897469	.3983969	.2871443	.2085517	.1635063	.1423725

N = 48

.1451302	.1569313	.1813862	.2205616	.2786875	.3671702
.4852100	1.1360700	1.1360700	.4852100	.3671702	.2786875
.2205616	.1813862	.1569319	.1451303	.1451302	.1569319
.1813862	.2205616	.2786875	.3671702	.4852100	1.1360700
1.1360700	.4852100	.3671702	.2786875	.2205616	.1813862
.1569319	.1451302	.1451303	.1569319	.1813862	.2205616
.2786876	.3671701	.4852100	1.1360700	1.1360700	.4852100
.3671702	.2786875	.2205615	.1813862	.1569319	.1451302

Table No. 46

CONJUGATE TORSION FUNCTION (ψ) FOR EQUILATERAL TRIANGLE

BY FIRST METHOD

Side length = 2

ordinates of the point P X	Y	Analytic value of ψ	Computed value of ψ for N = 12	Absolute error	Computed value of ψ for N = 24	Absolute error	Computed value of ψ for N = 36	Absolute error	Computed value of ψ for N = 48	Absolute error
69077	.0625	.1940141	.1960675	.0020534	.1945091	.0004950	.1941688	.0001547	.1940104	.0000037
69077	.1875	.2067089	.2158049	.0090960	.2071416	.0004327	.2067055	.0000034	.2067105	.0000016
69077	.3125	.2320984	.2453803	.0132819	.2329926	.0008942	.2320937	.0000047	.2321164	.0000180
69077	.4375	.2701827	.2893640	.0191813	.2709181	.0007354	.2705348	.0003521	.2702459	.0000632
69077	.5625	.3200318	.3466569	.0256951	.3240060	.0030442	.3217482	.0007864	.3211588	.0001970
69077	.6875	.3844357	.4241376	.0397019	.3927790	.0083433	.3861535	.0017178	.3850765	.0006408
69077	.8125	.4606043	.4427277	.0178766	.4770563	.0164520	.4672470	.0066427	.4636161	.0030118
60845	.0000	.2086587	.2151463	.0064876	.2093296	.0006709	.2087623	.0001036	.2086429	.0000158
60845	.1250	.2135416	.2216005	.0080589	.2143380	.0007964	.2136581	.0001165	.2135290	.0000126
60845	.2500	.2281900	.2401154	.0119254	.2290744	.0008844	.2283507	.0001607	.2281901	.0000001
60845	.3750	.2520041	.2683889	.0157848	.2539766	.0013725	.2528622	.0002581	.2526365	.0000324
60845	.5000	.2867839	.3082390	.0214551	.2890267	.0022428	.2872583	.0004744	.2868938	.0001099
60845	.6250	.3307292	.3585421	.0278129	.3355794	.0048502	.3317075	.0009783	.3310169	.00[illegible]877
60845	.7500	.3844403	.4241458	.0397055	.3927837	.0083434	.3861579	.0017176	.3850811	.0006408
52591	.0625	.2184244	.2273992	.0089748	.2192610	.0008366	.2185607	.0001363	.2184183	.0000061
52591	.1875	.2252603	.2358789	.0106186	.2261895	.0009292	.2254193	.0001590	.2252614	.0000011

Table No. 46 (CONTD.)

linates of point, P		Analytic value of ψ	Computed value of ψ for N = 12	Absolute error	Computed value of ψ for N = 24	Absolute error	Computed value of ψ for N = 36	Absolute error	Computed value of ψ for N = 48	Absolute error
[illegible]	Y									
25914	.3125	.2389323	.2522321	.0132998	.2400887	.0011564	.2391455	.0002132	.2389511	.0000188
25914	.4375	.2594401	.2762896	.0168495	.2610289	.0015888	.2597566	.0003165	.2594941	.0000540
25914	.5625	.2867839	.3082390	.0214551	.2890268	.0022429	.2872584	.0004745	.2868939	.0001100
25914	.6875	.3209637	.3466585	.0256948	.3240076	.0030439	.3217502	.0007865	.3211607	.0001970
43383	.0000	.2213541	.2309951	.0096410	.2222329	.0008788	.2215007	.0001466	.2213513	.0000028
43383	.1250	.2233072	.2333939	.0100867	.2242145	.0009073	.2234607	.0001535	.2233066	.0000006
43383	.2500	.2291666	.2405046	.0113380	.2301639	.0009973	.2293419	.0001753	.2291731	.0000065
43383	.3750	.2389323	.2522322	.0132999	.2400887	.0011564	.2391456	.0002133	.2389512	.0000189
43383	.5000	.2526043	.2683891	.0157848	.2539767	.0013724	.2528623	.0002580	.2526366	.0000323
43383	.6250	.2701825	.2893638	.0191813	.2709175	.0007350	.2705346	.0003521	.2702456	.0000631
60852	.0625	.2223307	.2321938	.0098631	.2232238	.0008931	.2224807	.0001500	.2223290	.0000017
60852	.1875	.2233073	.2333939	.0100866	.2242146	.0009073	.2234607	.0001534	.2233066	.0000007
60852	.3125	.2252604	.2358790	.0106186	.2261895	.0009291	.2254194	.0001590	.2252615	.0000011
60852	.4375	.2281902	.2401156	.0119254	.2290745	.0008843	.2283509	.0001607	.2281903	.0000001
60852	.5625	.2320965	.2453781	.0132816	.2329906	.0008941	.2320917	.0000048	.2321145	.0000180
21679	.0000	.2223307	.2321938	.0098631	.2232238	.0008931	.2224807	.0001500	.2223290	.0000017
21679	.1250	.2213541	.2309951	.0096410	.2222329	.0008788	.2215007	.0001466	.2213513	.0000028
21679	.2500	.2184245	.2273993	.0089748	.2192610	.0008365	.2185608	.0001363	.218418 4	.0000061
21679	.3750	.2135417	.2216007	.0080590	.2143382	.0007965	.2136583	.0001166	.2135292	.0000125
[illegible]679	.5000	.2067059	.2158018	.0090959	.2071384	.0004325	.2067024	.0000035	.2067075	.0000016

Table No. 46 (CONTD.)

inates of point P	Y	Analytic value of	Computed value of for N = 12	Absolute error	Computed value of for N = 24	Absolute error	Computed value of for N = 36	Absolute error	Computed value of for N = 48	Absolute error
4210	.0625	.2233072	.2333939	.0100867	.2242145	.0009073	.2234606	.0001534	.2233066	.0000006
4210	.1875	.2184244	.2273992	.0089748	.2192610	.0008366	.2185607	.0001363	.2184183	.0000061
4210	.3125	.2086589	.2151465	.0064876	.2093298	.0006709	.2087625	.0001036	.2086430	.0000159
4210	.4375	.1940105	.1960625	.0020520	.1945053	.0004948	.1941652	.0001547	.1940068	.0000037
36741	.0000	.2291665	.2405045	.0113380	.2301638	.0009973	.2293418	.0001753	.2291730	.0000065
36741	.1250	.2252603	.2358789	.0106186	.2261894	.0009291	.2254193	.0001590	.2252614	.0000011
36741	.2500	.2135416	.2216006	.0080590	.2143381	.0007965	.2136582	.0001166	.2135291	.0000125
36741	.3750	.1940104	.1960625	.0020521	.1945052	.0004948	.1941651	.0001547	.1940068	.0000037
59272	.0625	.2389321	.2522319	.0132998	.2400885	.0011564	.2391454	.0002133	.2389510	.0000189
59272	.1875	.2281899	.2401153	.0119254	.2290743	.0008844	.2283507	.0001608	.2281900	.0000001
69272	.3125	.2067056	.2158014	.0090958	.2071382	.0004326	.2067021	.0000035	.2067073	.0000017
51802	.0000	.2594398	.2762892	.0168494	.2610285	.0015887	.2597562	.0003164	.2594937	.0000539
51802	.1250	.2526039	.2683886	.0157847	.2539763	.0013724	.2528619	.0002580	.2526362	.0000323
51802	.2500	.2320961	.2453778	.0132817	.2329901	.0008940	.2320913	.0000048	.2321140	.0000179
34333	.0625	.2867834	.3082384	.0214550	.2890262	.0022428	.2872578	.0004744	.2868933	.0001099
34333	.1875	.2701819	.2893630	.0191811	.2709169	.0007350	.2705340	.0003521	.2702450	.0000631
16865	.0000	.3307285	.3585413	.0278128	.3355786	.0048501	.3317068	.0009783	.3310161	.0002876
16865	.1250	.3209629	.3466577	.0256948	.3240067	.0030438	.3217493	.0007864	.3211599	.0001970
99396	.0625	.3844392	.4241445	.0397053	.3927826	.0083434	.3861568	.0017176	.3850801	.0006409
21926	.0000	.4606108	.4427225	.0178883	.4770639	.0164531	.4672539	.0066431	.4636230	.0030122

Table No. 47

TORSION PROBLEM FOR EQUILATERAL TRIANGLE

Computed Values of μ

N = 12

.0997048	.5139147	.5139147	.0997048	.0997048	.5139147
.5139147	.0997048	.0997048	.5139146	.5139147	.0997048

N = 24

.0616342	.1681329	.3684782	.6366221	.6366221	.3684782
.1681729	.0616341	.0616342	.1681329	.3684782	.6366222
.6366221	.3684782	.1681329	.0616342	.0616341	.1681329
.3684782	.6366221	.6366221	.3684782	.1681329	.0616341

N = 36

.0545823	.1024658	.1958863	.3296841	.4944653	.6761150
.6761149	.4944653	.3296841	.1958863	.1024657	.0545823
.0545823	.1024657	.1958863	.3296842	.4944653	.6761141
.6761151	.4944653	.3296842	.1958863	.1024658	.0545823
.0545823	.1024657	.1958363	.3296841	.4944653	.6761149
.6761150	.4944653	.3296841	.1958863	.1024658	.0545823

N = 48

.0536005	.0805740	.1338360	.2118724	.3121810	.4308746
.5617047	.6959008	.6959477	.5617254	.4309866	.3123250
.2120555	.1340556	.0808411	.0539220	.0540038	.0810593
.1343330	.2122817	.3124009	.4308293	.5613521	.6952143
.6943490	.5598764	.4288283	.3099543	.2094905	.1313212
.0779530	-.0009744	.0509213	.0779821	.1313680	.2095520
.3100252	.4288999	.5599346	.6943732	.6950706	.5611206
.4305067	.3119288	.2118351	.1338750	.0806299	.0536259

Table No.48

CONJUGATE TORSION FUNCTION (ψ) FOR EQUILATERAL TRIANGLE

BY SECOND METHOD

Side length = 2

inates of point P X	Y	Analytic value of ψ	Computed value of ψ for N = 12	Absolute error	Computed value of ψ for N = 24	Absolute error	Computed value of ψ for N = 36	Absolute error	Computed value of ψ for N = 48	Absolute error
077	.0625	.1940141	.2096064	.0155923	.1964501	.0024360	.1949304	.0009163	.1943466	.0003325
077	.1875	.2067089	.2151568	.0084479	.2079497	.0012408	.2078191	.0011102	.2070662	.0003573
077	.3125	.2320984	.2302454	.0018530	.2355109	.0034125	.2328794	.0007810	.2324673	.0003689
077	.4375	.2701827	.2711869	.0010042	.2701297	.0000530	.2707010	.0005183	.2705430	.0003603
077	.5625	.3209618	.3520411	.0310793	.3250423	.0040805	.3218971	.0009353	.3212878	.0003260
077	.6875	.3844357	.4047020	.0202663	.3835430	.0008927	.3854011	.0009654	.3847015	.0002658
077	.8125	.4606043	.4387787	.0218256	.4690279	.0084236	.4605842	.0000201	.4608339	.0002296
0845	.0000	.2086587	.2205602	.0119015	.2108516	.0021929	.2095902	.0009315	.2087682	.0001095
0845	.1250	.2135416	.2237427	.0102011	.2155614	.0020198	.2144589	.0009173	.2137216	.0001800
0845	.2500	.2281900	.2348020	.0066120	.2303090	.0021190	.2290780	.0008880	.2284348	.0002448
0845	.3750	.2526041	.2587205	.0061164	.2544980	.0018939	.2534525	.0008484	.2528962	.0002921
0845	.5000	.2867839	.3006510	.0138671	.2887246	.0019407	.2875624	.0007785	.2870987	.0003148
0845	.6250	.3307292	.3524689	.0217397	.3324386	.0017094	.3314069	.0006777	.3310392	.0003100
0845	.7500	.3844403	.4047077	.0202674	.3835463	.0008940	.3854059	.0009656	.3847061	.0002658
2591	.0625	.2184244	.2283413	.0099169	.2205062	.0020818	.2193339	.0009095	.2183605	.0000639
2591	.1875	.2252603	.2340011	.0087408	.2273131	.0020528	.2261568	.0008965	.2253219	.0000616

CONTD..

Table No. 48 (CONTD.)

linates of point P X	Y	Analytic value of ψ	Computed value of ψ for N = 12	Absolute error	Computed value of ψ for N = 24	Absolute error	Computed value of ψ for N = 36	Absolute error	Computed value of ψ for N = 48	Absolute error
2591	.3125	.2389523	.2467857	.0078534	.2409493	.0020170	.2398021	.0008698	.2391044	.0001721
2591	.4375	.2594401	.2686609	.0092208	.2613689	.0019288	.2602696	.0008295	.2596977	.0002576
2591	.5625	.2867839	.3006511	.0138672	.2887247	.0019408	.2875625	.0007786	.2870977	.0003138
2591	.6875	.3209637	.3520478	.0310841	.3250453	.0040816	.3218992	.0009355	.3212892	.0003255
14338	.0000	.2213541	.2308609	.0095068	.2234251	.0020710	.2222580	.0009039	.2209779	.0003762
14338	.1250	.2233072	.2325373	.0092301	.2253703	.0020631	.2242074	.0009002	.2231329	.0001742
14338	.2500	.2291666	.2377747	.0086081	.2312117	.0020451	.2300554	.0008888	.2291775	.0000109
44338	.3750	.2389323	.2467858	.0078535	.2409494	.0020171	.2398022	.0008699	.2390983	.0001660
44338	.5000	.2526043	.2587207	.0061164	.2544982	.0018939	.2534526	.0008483	.2528914	.0002871
44338	.6250	.2701625	.2711834	.0010009	.2701285	.0000540	.2707006	.0005181	.2705407	.0003582
36085	.0625	.2223367	.2317008	.0093701	.2243977	.0020670	.2232327	.0009020	.2218501	.0004806
36085	.1875	.2233073	.2325373	.0092300	.2253703	.0020630	.2242074	.0009001	.2231092	.0001981
36085	.3125	.2252304	.2340011	.0087407	.2273131	.0020527	.2261569	.0008965	.2252988	.0000384
36085	.4375	.2281902	.2348021	.0066119	.2303092	.0021190	.2290781	.0008879	.2284202	.0002300
36085	.5625	.2330965	.2302414	.0018551	.2355096	.0034131	.2328775	.0007810	.2324595	.0003630
72167	.0000	.2223307	.2317008	.0093701	.2243977	.0020670	.2232327	.0009020	.2214511	.0008796
72167	.1250	.2213541	.2308609	.0095068	.2234252	.0020711	.2222580	.0009039	.2208954	.0004587
72167	.2500	.2184245	.2283413	.0099168	.2205063	.0020818	.2193339	.0009094	.2183007	.0001238
72167	.3750	.2135417	.2237428	.0102011	.2155615	.0020198	.2144590	.0009173	.2136870	.0001453
72167	.5000	.2067059	.2151539	.0084480	.2079463	.0012404	.2078162	.0011103	.2070494	.0003435

CONTD..

Table No. 1[illegible]

inates of point P		Analytic value of ψ	Computed value of ψ for N = 12	Absolute error	Computed value of ψ for N = 24	Absolute error	Computed value of ψ for N = 36	Absolute error	Computed value of ψ for N = 48	Absolute error
	Y									
421	.0625	.2273072	.2325373	.0092301	.2253702	.0020630	.2242073	.0009001	.2225467	.0007605
421	.1875	.2184244	.2283413	.0099169	.2205063	.0020819	.2193339	.0009095	.2181149	.0003095
421	.3125	.2086589	.2205603	.0119014	.2108518	.0021929	.2095903	.0009341	.2086992	.0000403
421	.4375	.1940105	.2096039	.0155934	.1964466	.0024361	.1949267	.0009162	.1943154	.0003049
3674	.0000	.2291665	.2377746	.0086081	.2312116	.0020451	.2300553	.0008888	.2291209	.0000456
8674	.1250	.2252603	.2340010	.0087407	.2273130	.0020527	.2261568	.0008965	.2247742	.0004861
8674	.2500	.2135416	.2237427	.0102011	.2155614	.0020198	.2144589	.0009173	.2134736	.0000680
8674	.3750	.1940104	.2096039	.0155935	.1964465	.0024361	.1949266	.0009162	.1942647	.0002543
6927	.0625	.2389321	.2407856	.0078535	.2409492	.0020171	.2398020	.0008699	.2383475	.0005846
6927	.1875	.2281899	.2348019	.0066120	.2303089	.0021190	.2290779	.0008880	.2280410	.0001489
6927	.3125	.2067056	.2151537	.0084481	.2079460	.0012404	.2078160	.0011104	.2069089	.0002033
5180	.0000	.2594398	.2686605	.0092207	.2613686	.0019288	.2602693	.0008295	.2589269	.0005129
5180	.1250	.2526039	.2587203	.0061164	.2544978	.0018939	.2534522	.0008483	.2524413	.0001626
5180	.2500	.2320961	.2302412	.0018549	.2355092	.0034131	.2328770	.0007809	.2322633	.0001672
13433	.0625	.2867834	.3006564	.0138670	.2887241	.0019407	.2875619	.0007785	.2866970	.0000864
13433	.1875	.2701819	.2711825	.0010006	.2701279	.0000540	.2707000	.0005181	.2703385	.0001566
21687	.0000	.3307285	.3524680	.0217395	.3324379	.0017094	.3314062	.0006777	.3307784	.0000499
21687	.1250	.3209629	.3520467	.0310838	.3250445	.0040816	.3218984	.0009355	.3211287	.0001658
29940	.0625	.3844392	.4047071	.0202679	.3835453	.0008939	.3854049	.0009657	.3846120	.0001727
38193	.0000	.4606108	.4387810	.0218298	.4690350	.0084242	.4605909	.0000199	.4608076	.0001968

CHAPTER 7

TORSION PROBLEM FOR GROOVED CIRCULAR SHAFT

The torsion problem for rectangular and equilateral triangular cross sections were done by numerical methods provided earlier. The results were compared from the analytical results which are available. The problem of notches, slots or grooves are technically important and in the next chapter two such problems, for which no solution is available, will be solved. To provide confidence in the results it appears necessary to solve a problem of a notch, where analytical solution is available. This relates to the problem of a notch done by C.Weber [50]. The boundary of the cross-section of the shaft is made up of the arcs of the two circles : $r = b$ and $r=2a \cos \theta$ (fig.13,pp.124). For attempting the problem numerically it is necessary to fix the values of a and b. Two cases where $a : b = 4 : 1$ and $a : b = 8 : 1$ were studied seperately. Both the methods were applied to solve these problems. We shall give only necessary steps of both the methods in one case only, where $a : b = 4 : 1$.

First Method -

The method followed to solve the problem is essentially the same as in the case of the problems done in the last chapter. But this time one has to be careful to provide enough points on the notch so as to approximate the curve properly by these

boundary points. Thus we have taken 2 and 4 points successively on the notch, and since we have taken the length of all the intervals to be equal, the total number of points turn out to be 18 and 36 respectively. It may however be mentioned that it is not necessary to take all intervals to be equal, but we have kept them equal in conformity with the method given for problems, done earlier.

As usual the values of σ_k are obtained from (100) which for the purpose of numerical computation in this case reduces to

$$\frac{1}{2}(x_i^2+y_i^2) = -\left[\sum_{k=1}^{8m}\sigma_k \int_{q_{k-1/2}}^{q_{k+1/2}} \log|q-q_i|\,dq + \sum_{k=8m+1}^{10m}\sigma_k \int_{q_{k-1/2}}^{q_{k+1/2}} \log|q'-q_i|\,dq' \right.$$

$$\left. + \sum_{k=10m+1}^{18m}\sigma_k \int_{q_{k-1/2}}^{q_{k+1/2}} \log|q-q_i|\,dq \right] \qquad \ldots (104)$$

where 2 m is the number of nodal points on the notch; (x_i, y_i) are coordinates of the nodal point $q_i, i=1,2,\ldots,N$. For convenience of numerical work we have replaced q by q' for a point if it lies on the notch. To evaluate the integrals in (104), approximation (42) is to be used. It may be noted here that the value of the radius of curvature in (42) is to be changed according to the position of the point q and that is why we have distinguished q' from q. Values of σ_k thus obtained are shown in Table No. 49, pp. 126. The value of Ψ at any point P inside the contour is computed after substituting the values of σ_k in (101). One half

of the cross-section about the axis of symmetry was covered by a square net of side .25 (fig.14,pp. 124) and Ψ was computed at each point of it. The analytic value of Ψ [50] was calculated at all these points from the following known expression.

$$\Psi\,(x,y) = a(x- \frac{b^2 x}{x^2+y^2}) + \frac{b^2}{2} \qquad \ldots (105)$$

The proper values of a and b are to be substituted in the above formula. The error at different points for values of N = 36 is shown in Table No.50,pp. 127. The maximum error at any point for N = 36 is about 18 % . Even though the value of Ψ was found to be fastly converging as N increases from 18 to 36, but still higher values of N could not be taken because of the limited capacity of the computer.

Second Method -

To make the comparative study further, of the two methods this problem was also solved by Second Method. In this case the values of μ_k are to be obtained from (102). These values appear in Table No. 51, pp. 126. The fixed line was taken as the tangent to the curve at the point p. Then to find at any point P, formula (103) was used. Value of Ψ was computed at all points of the cross-section mentioned earlier and shown in Table No.52,pp. 130. The maximum error at any point for N = 36 is 5.6 % . Here also Ψ converges fastly as N increases from 18 to 36. The stress function Ψ was also computed for N = 36 and a few lines of shearing stress are drawn as shown

in fig.15,pp. 125 .

Similar calculations by both the methods were done for another case where a : b = 8 : 1, to find out the effect on the accuracy of the methods as the size of the groove decreases. The values of σ_k and Ψ , obtained by First Method are given in Table Nos. 53 and 55, pps. 133, 134 respectively. The maximum error in Ψ at any point for N = 36 is 8.4 %. Similarly the values of μ_k and Ψ by Second Method are shown in Table Nos. 54 and 56, pps. 133, 137 respectively and the maximum error in Ψ at any point for N = 34 is 4.8 %. From the percentages of error by both the methods it seems that the accuracy improves as the size of the notch reduces. Apart from this the maximum error by Second Method remains less than that of First Method irrespective of the size of the notch. Entire computational work was done on the Russian Computer MINSK-2 where facility for doing computations only upto 7 significant digits is available. Autocode programmes for the First Method when a : b = 4 : 1 and for the Second Method when a : b = 8 : 1 are given in Appendix V.

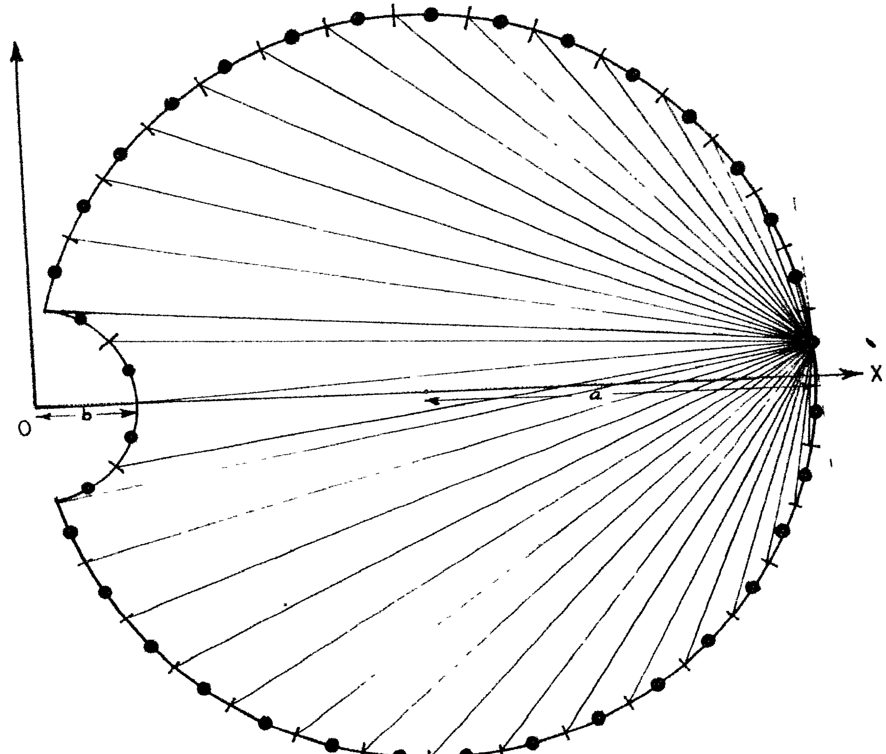

FIG. 13 - CIRCULAR CROSS-SECTION WITH A CIRCULAR NOTCH WHERE $a:b=4:1$ AND $N=36$

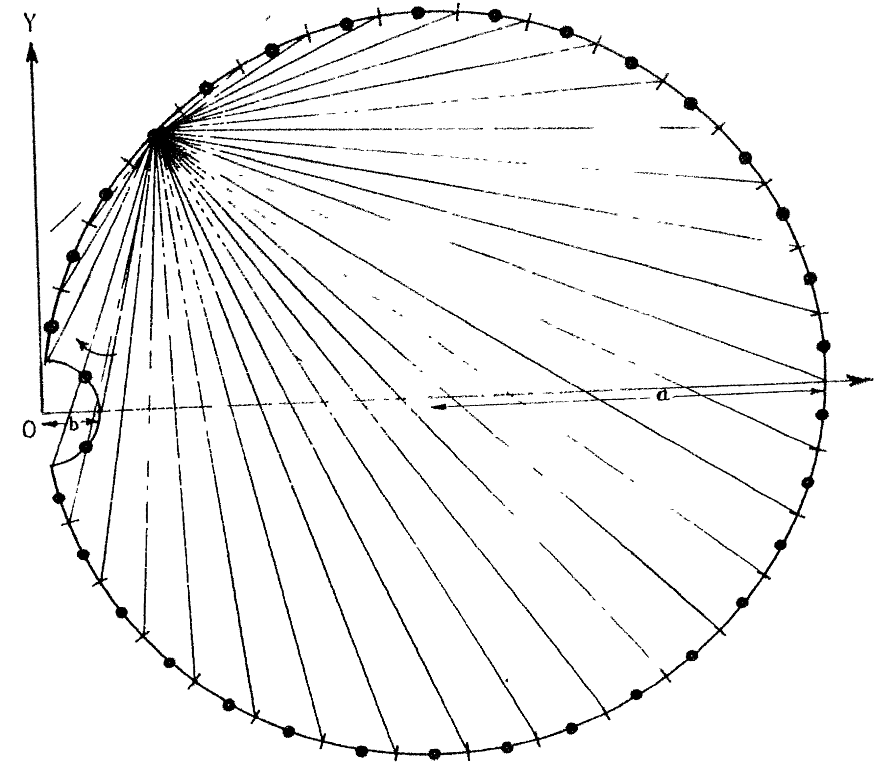

FIG. 14 - CIRCULAR CROSS-SECTION WITH A CIRCULAR NOTCH WHERE $a:b=8:1$ AND $N=34$

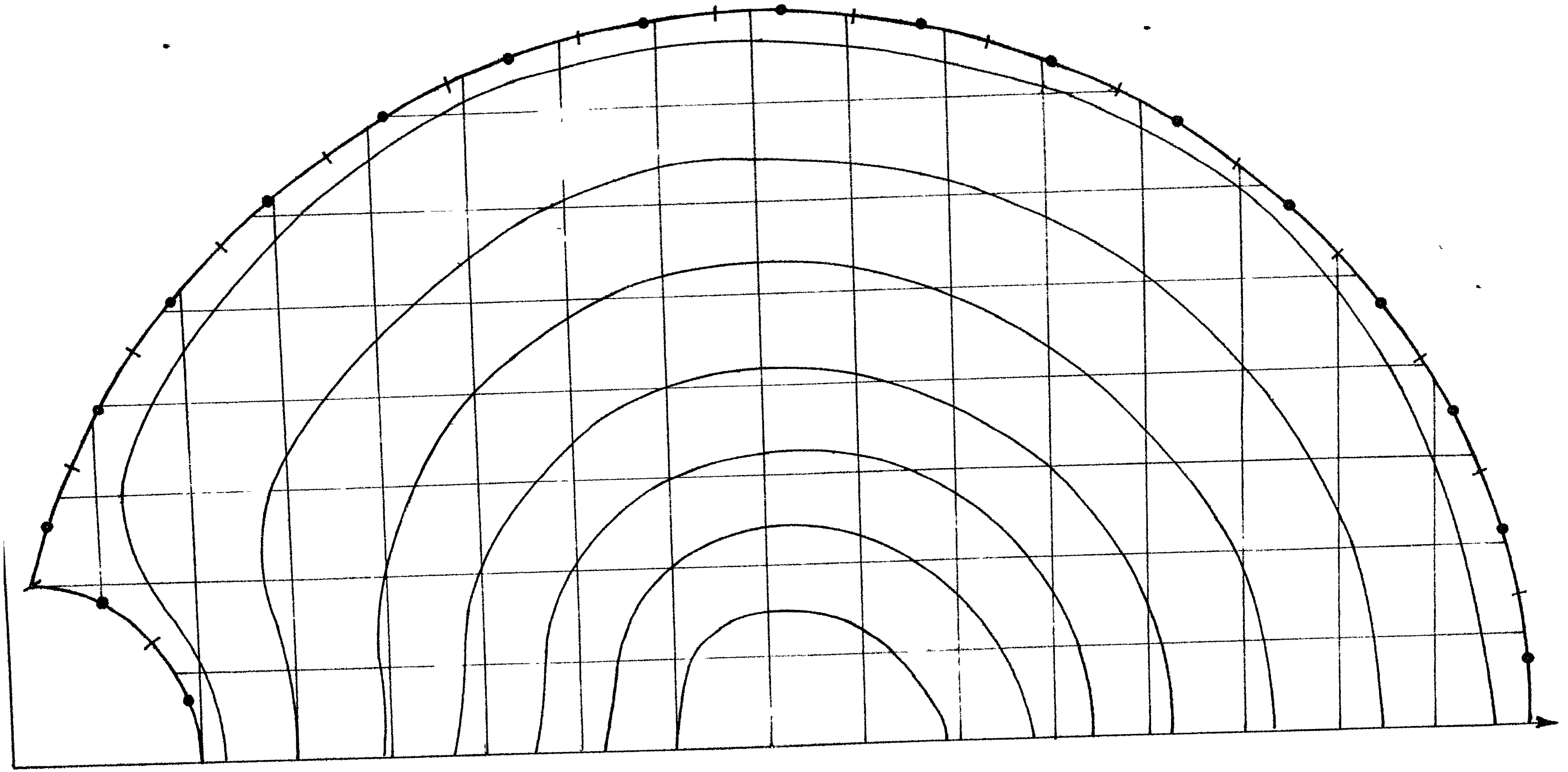

FIG. 15 – LINES OF SHEARING STRESS IN ONE HALF OF THE CIRCULAR CROSS-SECTION WITH A CIRCULAR NOTCH IN THE RATIO 4:1

TORSION PROBLEM FOR CIRCULAR CROSS-SECTION WITH A CIRCULAR NOTCH (a:b = 4:1)

Table No. 49

Computed Values of σ for N = 36

.1695262	.1489002	.1083235	.0491246	-.0267568	-.1168302
-.2181382	-.3273367	-.4408076	-.5547523	-.6652920	-.7685261
-.8604828	-.9376194	-.9850460	-1.1528440	-.7594455	-.7874400
-.7874404	-.7594451	-1.1528440	-.9850457	-.9376194	-.8604829
-.7685262	-.6652918	-.5547527	-.4408074	-.3273366	-.2181381
-.1168306	-.0267564	.0491245	.1083233	.1489002	.1695262

Table No. 51

Computed Values of μ for N = 36

12.084020	11.824530	11.313970	10.568970	9.613729	8.479304
7.202551	5.824923	4.391087	2.947430	1.540442	0.214934
-0.987734	-2.033110	-2.899886	-3.602048	-5.758908	-6.725574
-6.725574	-5.758989	-3.602048	-2.899886	-2.033110	-0.987734
0.214984	1.540442	2.947430	4.391087	5.824923	7.202552
8.479303	9.613730	10.568960	11.313980	11.824530	12.084020

Table No. 50

TORSION PROBLEM FOR CIRCULAR CROSS-SECTION WITH A CIRCULAR NOTCH (a:b = 4:1) FOR N = 36 BY FIRST METHOD

Coordinates of the point P		Analytic value of ψ	Computed value of ψ	Absolute error
X	Y			
1.25	0.00	2.2250000	2.2256830	0.0006830
1.50	0.00	2.7916660	2.7922040	0.0005380
1.75	0.00	3.3392850	3.3397480	0.0004630
2.00	0.00	3.8749990	3.8753890	0.0003900
2.25	0.00	4.4027770	4.4030780	0.0003010
2.50	0.00	4.9250000	4.9251950	0.0001950
2.75	0.00	5.4431810	5.4432570	0.0000758
3.00	0.00	5.9583330	5.9582760	0.0000570
3.25	0.00	6.4711530	6.4709540	0.0001990
3.50	0.00	6.9821420	6.9817910	0.0003510
3.75	0.00	7.4916660	7.4909480	0.0007180
1.00	0.25	1.6544110	1.6549330	0.0005220
1.25	0.25	2.2403840	2.2409380	0.0005540
1.50	0.25	2.8006750	2.8011910	0.0005160
1.75	0.25	3.3450000	3.3454650	0.0004650
2.00	0.25	3.8788460	3.8792430	0.0003970
2.25	0.25	4.4054870	4.4057960	0.0003090
2.50	0.25	4.9269800	4.9271830	0.0002030
2.75	0.25	5.4446720	5.4447540	0.0000820
3.00	0.25	5.9594820	5.9590000	0.0004820
3.25	0.25	6.4720580	6.4718640	0.0001940
3.50	0.25	6.9828680	6.9825200	0.0003480
3.75	0.25	7.4922560	7.4917040	0.0005520
1.00	0.50	1.7250000	1.7251000	0.0001000
1.25	0.50	2.2801720	2.2806020	0.0004300
1.50	0.50	2.8250000	2.8255060	0.0005060
1.75	0.50	3.3608490	3.3613370	0.0004880
2.00	0.50	3.8897050	3.8901310	0.0004260
2.25	0.50	4.4132350	4.4135700	0.0003350
2.50	0.50	4.9326920	4.9329170	0.0002250
2.75	0.50	5.4490000	5.4491000	0.0001000

CONTD...

Table No. 50 (CONTD.)

3.00	0.50	5.9628370	5.9628010	0.0000360
3.25	0.50	6.4747100	6.4745280	0.0001820
3.50	0.50	6.9849990	6.9846640	0.0003350
3.75	0.50	7.4939950	7.4942400	0.0002450
0.75	0.75	1.2916660	1.2915300	0.0001360
1.00	0.75	1.8050000	1.8053040	0.0003040
1.25	0.75	2.3308820	2.3313980	0.0005160
1.50	0.75	2.8583330	2.8589090	0.0005760
1.75	0.75	3.3836200	3.3841710	0.0005510
2.00	0.75	3.9058210	3.9063010	0.0004800
2.25	0.75	4.4250000	4.4253800	0.0003800
2.50	0.75	4.9415130	4.9417760	0.0002630
2.75	0.75	5.4557690	5.4559000	0.0001310
3.00	0.75	5.9681370	5.9681260	0.0000110
3.25	0.75	6.4789320	6.4787700	0.0001620
3.50	0.75	6.9884140	6.9881300	0.0002840
3.75	0.75	7.4967940	7.4950500	0.0017440
0.50	1.00	0.9250000	0.9278946	0.0028946
0.75	1.00	1.3849990	1.3856660	0.0006670
1.00	1.00	1.8750000	1.8757330	0.0007330
1.25	1.00	2.3810970	2.3818500	0.0007530
1.50	1.00	2.8942300	2.8949560	0.0007250
1.75	1.00	3.4096150	3.4102720	0.0006570
2.00	1.00	3.9250000	3.9255600	0.0005600
2.25	1.00	4.4394320	4.4398760	0.0004440
2.50	1.00	4.9525860	4.9528990	0.0003130
2.75	1.00	5.4644160	5.4645870	0.0001710
3.00	1.00	5.9750000	5.9750210	0.0000210
3.25	1.00	6.4844590	6.4843300	0.0001290
3.50	1.00	6.9929240	6.9924710	0.0004530
0.50	1.25	0.9870689	0.9109175	0.0671514
0.75	1.25	1.4485290	1.4513980	0.0028690
1.00	1.25	1.9298780	1.9310280	0.0011500
1.25	1.25	2.4249990	2.4260400	0.0010410
1.50	1.25	2.9282780	2.9291940	0.0009160

Table No.50 (CONTD.)

1.75	1.25	3.4358100	3.4366020	0.0007920
2.00	1.25	3.9452240	3.9458850	0.0006610
2.25	1.25	4.4551880	4.4557090	0.0005210
2.50	1.25	4.9650000	4.9653730	0.0003730
2.75	1.25	5.4743150	5.4745360	0.0002210
3.00	1.25	5.9829880	5.9830540	0.0000660
3.25	1.25	6.4909790	6.4907760	0.0002030
3.50	1.25	6.9983030	6.9974260	0.0008770
0.75	1.50	1.4916660	1.4205370	0.0711290
1.00	1.50	1.9711530	1.9752180	0.0040650
1.25	1.50	2.4610650	2.4623300	0.0012650
1.50	1.50	2.9583330	2.9594170	0.0010840
1.75	1.50	3.4602940	3.4612250	0.0009310
2.00	1.50	3.9650000	3.9657680	0.0007680
2.25	1.50	4.4711530	4.4717610	0.0006080
2.50	1.50	4.9779410	4.9783920	0.0004510
2.75	1.50	5.4848720	5.4851570	0.0002850
3.00	1.50	5.9916660	5.9913500	0.0003160
3.25	1.50	6.4981700	6.5008820	0.0027120
1.25	1.75	2.4898640	2.5030560	0.0131920
1.50	1.75	2.9838230	2.9852190	0.0013960
1.75	1.75	3.4821420	3.4822670	0.0001250
2.00	1.75	3.9834070	3.9842150	0.0008080
2.25	1.75	4.4865380	4.4878220	0.0012840
2.50	1.75	4.9907710	4.9917840	0.0010130
2.75	1.75	5.4955880	5.4900240	0.0055640
0.75	0.00	0.9583333	0.9638432	0.0055099
0.50	0.25	0.3249999	0.3056583	0.0193416
0.75	0.25	1.0250000	1.0238220	0.0011780
0.25	0.50	0.2249999	0.1825041	0.0424958
0.50	0.50	0.6250000	0.6251131	0.0001131
0.75	0.50	1.1634610	1.1627950	0.0006660
0.25	0.75	0.4250000	0.4341054	0.0091054
0.50	0.75	0.8173076	0.8161102	0.0011974
1.00	0.00	1.6249990	1.6262710	0.0012720

Table No. 52

TORSION PROBLEM FOR CIRCULAR CROSS-SECTION WITH A CIRCULAR NOTCH (a:b = 4:1) FOR N = 36 BY SECOND METHOD

Coordinates of the point P X	Coordinates of the point P Y	Analytic value of ψ	Computed value of ψ	Absolute error
1.25	0.00	2.2249990	2.2327660	0.0077670
1.50	0.00	2.7916660	2.7973050	0.0056390
1.75	0.00	3.3392850	3.3432470	0.0039620
2.00	0.00	3.8749990	3.8775440	0.0025450
2.25	0.00	4.4027770	4.4040720	0.0012950
2.50	0.00	4.9250000	4.9251610	0.0001610
2.75	0.00	5.4431810	5.4422920	0.0008890
3.00	0.00	5.9583330	5.9564560	0.0018770
3.25	0.00	6.4711530	6.4683360	0.0028170
3.50	0.00	6.9821420	6.9784190	0.0037230
3.75	0.00	7.4916660	7.4867110	0.0049550
1.00	0.25	1.6544110	1.6641290	0.0097180
1.25	0.25	2.2403840	2.2477420	0.0073580
1.50	0.25	2.8006750	2.8061090	0.0054340
1.75	0.25	3.3450000	3.3488380	0.0038380
2.00	0.25	3.8788460	3.8813090	0.0024630
2.25	0.25	4.4054870	4.4067260	0.0012390
2.50	0.25	4.9269800	4.9271000	0.0001200
2.75	0.25	5.4446720	5.4437520	0.0009200
3.00	0.25	5.9594820	5.9575810	0.0019010
3.25	0.25	6.4720580	6.4692220	0.0028360
3.50	0.25	6.9828680	6.9791300	0.0037378
3.75	0.25	7.4922560	7.4879890	0.0042670
1.00	0.50	1.7250000	1.7331090	0.0081090
1.25	0.50	2.2801720	2.2866370	0.0064650
1.50	0.50	2.8250000	2.8299070	0.0049070
1.75	0.50	3.3608490	3.3643520	0.0035030
2.00	0.50	3.8897050	3.8919420	0.0022370
2.25	0.50	4.4132350	4.4143130	0.0010780
2.50	0.50	4.9326920	4.9326940	0.0000020
2.75	0.50	5.4490000	5.4479910	0.0010090

Table No. 52 (CONTD.)

3.00	0.50	5.9628370	5.9608680	0.0019690
3.25	0.50	6.4747100	6.4718200	0.0028900
3.50	0.50	6.9849990	6.9812390	0.0037600
3.75	0.50	7.4939950	7.4909140	0.0030810
0.75	0.75	1.2916660	1.2996700	0.0080040
1.00	0.75	1.8050000	1.8116830	0.0066830
1.25	0.75	2.3308820	2.3363260	0.0054440
1.50	0.75	2.8583330	2.8625490	0.0042160
1.75	0.75	3.3836200	3.3866500	0.0030300
2.00	0.75	3.9058210	3.9077240	0.0019030
2.25	0.75	4.4250000	4.4258350	0.0008350
2.50	0.75	4.9415130	4.9413340	0.0001790
2.75	0.75	5.4557690	5.4546210	0.0011480
3.00	0.75	5.9681370	5.9660590	0.0020770
3.25	0.75	6.4789320	6.4759550	0.0029770
3.50	0.75	6.9884140	6.9847440	0.0036700
3.75	0.75	7.4967940	7.4757770	0.0210170
0.50	1.00	0.9250000	0.9278674	0.0028674
0.75	1.00	1.3849990	1.3907520	0.0057520
1.00	1.00	1.8750000	1.8802370	0.0052370
1.25	1.00	2.3810970	2.3855190	0.0044220
1.50	1.00	2.8942300	2.8977100	0.0034800
1.75	1.00	3.4096150	3.4121100	0.0024950
2.00	1.00	3.9250000	3.9265070	0.0015070
2.25	1.00	4.4394320	4.4399710	0.0005390
2.50	1.00	4.9525860	4.9521800	0.0004060
2.75	1.00	5.4644160	5.4630910	0.0013250
3.00	1.00	5.9750000	5.9727810	0.0022190
3.25	1.00	6.4844590	6.4814360	0.0030230
3.50	1.00	6.9929240	6.9855370	0.0073870
0.50	1.25	0.9870689	1.0000160	0.0129471
0.75	1.25	1.4485290	1.4525370	0.0040080
1.00	1.25	1.9298780	1.9338920	0.0040140
1.25	1.25	2.4249990	2.4285140	0.0035150
1.50	1.25	2.9282780	2.9310590	0.0027810
1.75	1.25	3.4358100	3.4377680	0.0019580

Table No. 52 (CONTD.)

2.00	1.25	3.9452240	3.9463180	0.0010940
2.25	1.25	4.4551880	4.4554040	0.0002160
2.50	1.25	4.9650000	4.9643400	0.0006600
2.75	1.25	5.4743150	5.4727920	0.0015230
3.00	1.25	5.9829880	5.9806130	0.0023750
3.25	1.25	6.4909790	6.4861220	0.0048570
3.50	1.25	6.9983030	7.0577200	0.0594170
0.75	1.50	1.4916660	1.5054460	0.0137800
1.00	1.50	1.9711530	1.9765200	0.0053670
1.25	1.50	2.4610650	2.4633730	0.0023080
1.50	1.50	2.9583330	2.9604830	0.0021500
1.75	1.50	3.4602940	3.4617620	0.0014680
2.00	1.50	3.9650000	3.9657000	0.0007000
2.25	1.50	4.4711530	4.4710610	0.0000920
2.50	1.50	4.9779410	4.9770470	0.0008940
2.75	1.50	5.4848720	5.4827680	0.0021040
3.00	1.50	5.9916660	5.9876370	0.0040290
3.25	1.50	6.4981700	6.5712290	0.0730590
1.25	1.75	2.4989640	2.5116210	0.0217570
1.50	1.75	2.9838230	2.9771050	0.0067180
1.75	1.75	3.4821420	3.4826830	0.0005410
2.00	1.75	3.9834070	3.9858230	0.0024160
2.25	1.75	4.4865380	4.4877960	0.0012580
2.50	1.75	4.9907710	4.9823800	0.0083910
2.75	1.75	5.4955880	5.4852330	0.0103550
0.75	0.00	0.9583333	0.9766523	0.0183190
0.50	0.25	0.3249999	0.3062893	0.0187106
0.75	0.25	1.0250000	1.0363600	0.0113600
0.25	0.50	0.2249999	0.2187278	0.0062721
0.50	0.50	0.6250000	0.6451393	0.0201393
0.75	0.50	1.1634610	1.1732300	0.0097690
0.25	0.75	0.4250000	0.4105112	0.0144888
0.50	0.75	0.8173076	0.8261384	0.0088308
1.00	0.00	1.6249990	1.6359180	0.0109190

TORSION PROBLEM FOR CIRCULAR CROSS-SECTION WITH A CIRCULAR NOTCH (a:b = 8:1)

Table No. 53

Computed Values of σ for N = 34

.1737304	.1512557	.1071037	.0428417	-.0392496	-.1362546
-.2447270	-.3608080	-.4803625	-.5991158	-.7127904	-.8172270
-.9084095	-.9825481	-1.0302770	-1.0931100	-.7564157	-.7564157
-1.0931100	-1.0302770	-.9825482	-.9084096	-.8172270	-.7127905
-.5991153	-.4803627	-.3608082	-.2447270	-.1362545	-.0392494
.0428414	.1071040	.1512555	.1737304		

Table No. 54

Computed Values of μ for N = 34

11.992570	11.710310	11.155810	10.348700	9.317601	8.099042
6.736212	5.277407	3.774330	2.280244	0.848101	-0.471346
-1.631337	-2.590763	-3.315623	-3.780228	-5.373866	-5.373866
-3.780228	-3.315623	-2.590763	-1.631337	-0.471346	0.848101
2.280244	3.774329	5.277407	6.736211	8.099042	9.317600
10.348700	11.155810	11.710310	11.992570		

Table No. 55

TORSION PROBLEM FOR CIRCULAR CROSS-SECTION WITH A CIRCULAR NOTCH (a:b = 8:1) FOR N = 34 BY FIRST METHOD

Coordinates of the point P		Analytic value of ψ	Computed value of ψ	Absolute error
X	Y			
1.25	0.00	2.4312500	2.4346610	0.0034110
1.50	0.00	2.9479160	2.9506230	0.0027070
1.75	0.00	3.4598210	3.4619950	0.0021740
2.00	0.00	3.9687500	3.9704820	0.0017320
2.25	0.00	4.4756940	4.4770420	0.0013480
2.50	0.00	4.9812490	4.9822480	0.0009990
2.75	0.00	5.4857950	5.4864700	0.0006750
3.00	0.00	5.9895830	5.9899530	0.0003700
3.25	0.00	6.4927880	6.4928660	0.0000780
3.50	0.00	6.9955350	6.9953310	0.0002040
3.75	0.00	7.4979160	7.4971690	0.0007470
1.00	0.25	1.9136020	1.9176830	0.0040810
1.25	0.25	2.4350960	2.4383450	0.0032490
1.50	0.25	2.9501680	2.9527970	0.0026290
1.75	0.25	3.4612500	3.4633800	0.0021300
2.00	0.25	3.9697110	3.9714170	0.0017060
2.25	0.25	4.4763710	4.4777020	0.0013310
2.50	0.25	4.9817450	4.9827310	0.0009860
2.75	0.25	5.4861680	5.4868340	0.0006660
3.00	0.25	5.9898700	5.9902330	0.0003630
3.25	0.25	6.4930140	6.4930870	0.0000730
3.50	0.25	6.9957170	6.9955080	0.0002090
3.75	0.25	7.4980640	7.4975980	0.0004660
1.00	0.50	1.9312500	1.9345660	0.0033160
1.25	0.50	2.4450430	2.4479300	0.0028870
1.50	0.50	2.9562490	2.9586850	0.0024360
1.75	0.50	3.4652120	3.4672280	0.0020160
2.00	0.50	3.9724260	3.9740600	0.0016340
2.25	0.50	4.4783080	4.4795890	0.0012810
2.50	0.50	4.9831730	4.9841240	0.0009510
[illegible]	[illegible]	5.4872500	5.4878900	0.0006400

Table No. 55 (CONTD.)

3.00	0.50	5.9907090	5.9910520	0.0003430
3.25	0.50	6.4936770	6.4937350	0.0000580
3.50	0.50	6.9962500	6.9960350	0.0002150
3.75	0.50	7.4984980	7.4989000	0.0004020
0.75	0.75	1.4479160	1.4506500	0.0027340
1.00	0.75	1.9512500	1.9539940	0.0027440
1.25	0.75	2.4577200	2.4602400	0.0025200
1.50	0.75	2.9645830	2.9667870	0.0022040
1.75	0.75	3.4709050	3.4727700	0.0018650
2.00	0.75	3.9764550	3.9779860	0.0015310
2.25	0.75	4.4812490	4.4824580	0.0012090
2.50	0.75	4.9853780	4.9862760	0.0008980
2.75	0.75	5.4889420	5.4895420	0.0006000
3.00	0.75	5.9920340	5.9923460	0.0003120
3.25	0.75	6.4947330	6.4947660	0.0000330
3.50	0.75	6.9971030	6.9969020	0.0002010
3.75	0.75	7.4991980	7.4962870	0.0029110
0.50	1.00	0.9812500	0.9798631	0.0013869
0.75	1.00	1.4712500	1.4737720	0.0025220
1.00	1.00	1.9687500	1.9711490	0.0023990
1.25	1.00	2.4702740	2.4725040	0.0022300
1.50	1.00	2.9735570	2.9755440	0.0019870
1.75	1.00	3.4774030	3.4791110	0.0017080
2.00	1.00	3.9812500	3.9826650	0.0014150
2.25	1.00	4.4848580	4.4859800	0.0011220
2.50	1.00	4.9881460	4.9889800	0.0008340
2.75	1.00	5.4911040	5.4916540	0.0005500
3.00	1.00	5.9937500	5.9940220	0.0002720
3.25	1.00	6.4961140	6.4961240	0.0000100
3.50	1.00	6.9982310	6.9976240	0.0006070
0.50	1.25	0.9967672	1.0177780	0.0210108
0.75	1.25	1.4871320	1.4874860	0.0003540
1.00	1.25	1.9824690	1.9847050	0.0022360
1.25	1.25	2.4812500	2.4832630	0.0020130
1.50	1.25	2.9820690	2.9838720	0.0018030
1.75	1.25	3.4839520	3.4855130	0.0015610

Table No. 55 (CONTD.)

2.00	1.25	3.9863060	3.9876060	0.0013000
2.25	1.25	4.4887970	4.4898290	0.0010320
2.50	1.25	4.9912500	4.9920120	0.0007620
2.75	1.25	5.4935780	5.4940720	0.0004940
3.00	1.25	5.9957470	5.9959840	0.0002370
3.25	1.25	6.4977440	6.4975910	0.0001530
3.50	1.25	6.9995750	7.0006790	0.0011040
0.75	1.50	1.4979160	1.5167040	0.0187880
1.00	1.50	1.9927880	1.9915050	0.0012830
1.25	1.50	2.4902660	2.4922210	0.0019550
1.50	1.50	2.9895830	2.9912640	0.0016810
1.75	1.50	3.4900730	3.4915030	0.0014300
2.00	1.50	3.9912500	3.9924390	0.0011890
2.25	1.50	4.4927880	4.4937270	0.0009390
2.50	1.50	4.9944850	4.9951720	0.0006870
2.75	1.50	5.4962180	5.4967240	0.0005060
3.00	1.50	5.9979160	5.9979370	0.0000210
3.25	1.50	6.4995420	6.5005100	0.0009680
0.25	0.25	0.2812500	0.2575357	0.0237143
1.25	1.75	2.4974660	2.4765770	0.0208890
1.50	1.75	2.9959550	2.9992470	0.0032920
1.75	1.75	3.4955350	3.4977200	0.0021850
2.00	1.75	3.9958510	3.9964840	0.0006330
2.25	1.75	4.4966340	4.4969180	0.0002840
2.50	1.75	4.9976920	4.9995870	0.0018950
2.75	1.75	5.4988970	5.5014070	0.0025100
0.50	0.00	0.7812500	0.7952787	0.0141287
0.75	0.00	1.3645830	1.3712850	0.0067020
0.50	0.25	0.8312500	0.8367188	0.0054688
0.75	0.25	1.3812500	1.3864660	0.0052160
0.25	0.50	0.4312500	0.4345592	0.0033092
0.50	0.50	0.9062500	0.9086489	0.0023989
0.75	0.50	1.4158650	1.4193060	0.0034410
0.25	0.75	0.4812500	0.4504682	0.0307818
0.50	0.75	0.9543269	0.9567820	0.0024551
1.00	0.00	1.9062500	1.9107480	0.0044980

Table No. 56

TORSION PROBLEM FOR CIRCULAR CROSS-SECTION WITH A CIRCULAR NOTCH (a:b = 8:1) FOR N = 34 BY SECOND METHOD

Coordinates of the point P: X	Coordinates of the point P: Y	Analytic value of ψ	Computed value of ψ	Absolute error
1.25	0.00	2.4312500	2.4407290	0.0094790
1.50	0.00	2.9479160	2.9547730	0.0068570
1.75	0.00	3.4598210	3.4646560	0.0048350
2.00	0.00	3.9687500	3.9719110	0.0031610
2.25	0.00	4.4756940	4.4774010	0.0017070
2.50	0.00	4.9812490	4.9816520	0.0004030
2.75	0.00	5.4857950	5.4849980	0.0007970
3.00	0.00	5.9895830	5.9876650	0.0019180
3.25	0.00	6.4927880	6.4798080	0.0029800
3.50	0.00	6.9955350	6.9915350	0.0040000
3.75	0.00	7.4979160	7.4924800	0.0054360
1.00	0.25	1.9136020	1.9257070	0.0121050
1.25	0.25	2.4350960	2.4440470	0.0089510
1.50	0.25	2.9501680	2.9567470	0.0065790
1.75	0.25	3.4612500	3.4659200	0.0046700
2.00	0.25	3.9697110	3.9727660	0.0030550
2.25	0.25	4.4763710	4.4780060	0.0016350
2.50	0.25	4.9817450	4.9820950	0.0003500
2.75	0.25	5.4861680	5.4853320	0.0008360
3.00	0.25	5.9898700	5.9879230	0.0019470
3.25	0.25	6.4930140	6.4900110	0.0030030
3.50	0.25	6.9957170	6.9917010	0.0040160
3.75	0.25	7.4980640	7.4936310	0.0044330
1.00	0.50	1.9312500	1.9408700	0.0096200
1.25	0.50	2.4450430	2.4527260	0.0076830
1.50	0.50	2.9562490	2.9621070	0.0058580
1.75	0.50	3.4652120	3.4694370	0.0042250
2.00	0.50	3.9724260	3.9751870	0.0027610
2.25	0.50	4.4783080	4.4797380	0.0014300
2.50	0.50	4.9831730	4.9833750	0.0002020
[illegible]	[illegible]	5.4872500	5.4863030	0.0009470

CONTD...

Table No. 56 (CONTD.)

3.00	0.50	5.9907090	5.9886760	0.0020330
3.25	0.50	6.4936770	6.4906070	0.0030700
3.50	0.50	6.9962500	6.9922190	0.0040310
3.75	0.50	7.4984980	7.4942610	0.0042370
0.75	0.75	1.4479160	1.4556910	0.0077750
1.00	0.75	1.9512500	1.9585960	0.0073460
1.25	0.75	2.4577200	2.4639650	0.0062450
1.50	0.75	2.9645830	2.9695170	0.0049340
1.75	0.75	3.4709050	3.4745140	0.0036090
2.00	0.75	3.9764550	3.9787890	0.0023340
2.25	0.75	4.4812490	4.4823730	0.0011240
2.50	0.75	4.9853780	4.9853530	0.0000250
2.75	0.75	5.4889420	5.4878230	0.0011190
3.00	0.75	5.9920340	5.9898670	0.0021670
3.25	0.75	6.4947330	6.4915580	0.0031750
3.50	0.75	6.9971030	6.9929940	0.0041090
3.75	0.75	7.4991980	7.4757050	0.0234930
0.50	1.00	0.9812500	0.9872078	0.0059578
0.75	1.00	1.4712500	1.4771580	0.0059080
1.00	1.00	1.9687500	1.9744290	0.0056790
1.25	1.00	2.4702740	2.4752440	0.0049700
1.50	1.00	2.9735570	2.9775630	0.0040060
1.75	1.00	3.4774030	3.4803400	0.0029370
2.00	1.00	3.9812500	3.9830900	0.0018400
2.25	1.00	4.4848580	4.4856130	0.0007550
2.50	1.00	4.9881460	4.9878400	0.0003060
2.75	1.00	5.4911040	5.4897660	0.0013380
3.00	1.00	5.9937500	5.9914080	0.0023420
3.25	1.00	6.4961140	6.4928690	0.0032450
3.50	1.00	6.9982310	6.9904140	0.0078170
0.50	1.25	0.9967672	1.0235470	0.0267802
0.75	1.25	1.4871320	1.4914010	0.0042690
1.00	1.25	1.9824690	1.9870330	0.0045640
1.25	1.25	2.4812500	2.4851780	0.0039280
1.50	1.25	2.9820690	2.9852400	0.0031710
1.75	1.25	3.4839520	3.4862360	0.0022840

Table No. 56 (CONTD.)

2.00	1.25	3.9863060	3.9876390	0.0013330
2.25	1.25	4.4887970	4.4891560	0.0003590
2.50	1.25	4.9912500	4.9906320	0.0006180
2.75	1.25	5.4935780	5.4919930	0.0015850
3.00	1.25	5.9957470	5.9933230	0.0024240
3.25	1.25	6.4977440	6.4916110	0.0061330
3.50	1.25	6.9995750	7.0490480	0.0494730
0.75	1.50	1.4979160	1.4906440	0.0072720
1.00	1.50	1.9927880	1.9939770	0.0011890
1.25	1.50	2.4902660	2.4939900	0.0037240
1.50	1.50	2.9895830	2.9920260	0.0024430
1.75	1.50	3.4900730	3.4917370	0.0016640
2.00	1.50	3.9912500	3.9920820	0.0008320
2.25	1.50	4.4927880	4.4927460	0.0000420
2.50	1.50	4.9944850	4.9936180	0.0008670
2.75	1.50	5.4962180	5.4947720	0.0014460
3.00	1.50	5.9979160	5.9888400	0.0090760
3.25	1.50	6.4995420	6.5488440	0.0493020
0.25	0.25	0.2812500	0.2805326	0.0007174
1.25	1.75	2.4974660	2.4939200	0.0035460
1.50	1.75	2.9959550	3.0074520	0.0114970
1.75	1.75	3.4955350	3.4947610	0.0007740
2.00	1.75	3.9958510	3.9937940	0.0020570
2.25	1.75	4.4966340	4.4981420	0.0015080
2.50	1.75	4.9976920	5.0043370	0.0066450
2.75	1.75	5.4988970	5.4542350	0.0446620
0.50	0.00	0.7812500	0.8188047	0.0375547
0.75	0.00	1.3645830	1.3846710	0.0200880
0.50	0.25	0.8312500	0.8529419	0.0216919
0.75	0.25	1.3812500	1.3978010	0.0165510
0.25	0.50	0.4312500	0.4302496	0.0010004
0.50	0.50	0.9062500	0.9158632	0.0096132
0.75	0.50	1.4158650	1.4269520	0.0110870
0.25	0.75	0.4812500	0.4850455	0.0037955
0.50	0.75	0.9543269	0.9609214	0.0065945
1.00	0.00	1.9062500	1.9195510	0.0133010

CHAPTER 8

TORSION PROBLEM FOR NOTCHED RECTANGULAR CROSS-SECTION

In this chapter two torsion problems, for which the analytical solutions do not seem to exist are solved. Both problems were done by First and Second Methods. It is thought that the values of conjugate torsion function Ψ in a given actual case will be similar to those obtained here.

We have taken a rectangular cross-section of sides 2 X 1. As a first example the notch is on one of the larger sides, is symmetrically situated and is rectangular in shape. The size of the notch is .4 X .2 as shown in fig. 16,pp. 143. The coordinate system is also shown in the same figure. Two checks were employed to ascertain the value of Ψ . First the problem was done by First Method, taking 16,32 and 48 nodal points successively on the boundary. The values of Ψ obtained are given in Table No. 59,pp. 150 . It appears that the results are convergent and perhaps the true value of Ψ is near to the value obtained for N = 48. The same results were then obtained by Second Method for the same values of N and are shown in Table No. 60, pp. 153 . It turns out that the values of Ψ by First and Second Method for N = 48 are almost identical in first two places as shown in fig. 18,pp. 144 . Maximum difference being about 2 % . This is also attributed for small number of nodal points and also to the computer available, where computation only upto seven significant figures could be done.

The lines of shearing stress are also drawn in this case and are shown in fig. 19, pp. 145. The stresses τ_{zx} and τ_{zy} can be computed from the following formulae :

$$\tau_{zx} = \mu\alpha\left(\frac{\partial\Psi}{\partial y} - y\right) \quad , \quad \tau_{yz} = \mu\alpha\left(-\frac{\partial\Psi}{\partial x} + x\right) \qquad \ldots (107)$$

where the values of $\frac{\partial\Psi}{\partial y}$ and $\frac{\partial\Psi}{\partial x}$ in the above expressions are to be approximated by its proper difference formulae. The value of maximum shearing stress τ is given by

$$\tau = (\tau_{zx}^2 + \tau_{zy}^2)^{1/2} \qquad \ldots (108)$$

These values were also computed for N = 48 at all grid points shown in fig.17,pp. 143 and are given in Table Nos. 59 and 60, pps. 150, 153.

As a second example we have again taken the rectangular cross-section of sides 2 X 1 with a symmetrically situated equilateral triangular notch of depth $.2\sqrt{3}$, on one of the larger sides as shown in fig. 20, pp. 146 . The same calculations were done as for the case of the rectangular notch by taking the same values of N and are given in Table Nos. 63 and 64, pps. 157, 160. The values of Ψ for N = 48, by both the methods at all grid points are shown in fig.22,pp. 147 . The maximum difference between them at any point is about 4 % , and perhaps are nearer to the exact value. The lines of shearing stress are also drawn in this case and are shown in fig. 23, pp. 148.

Both problems were solved using Russian computer MINSK-2.

A single auto code programme each for computing the value of stress function Ψ by First Method in case of the rectangular notch and by Second Method in case of the triangular notch is given in Appendix VI. Another two programmes for computing the value of maximum shearing stress τ by any of the two methods in both the cases are also given in the same Appendix.

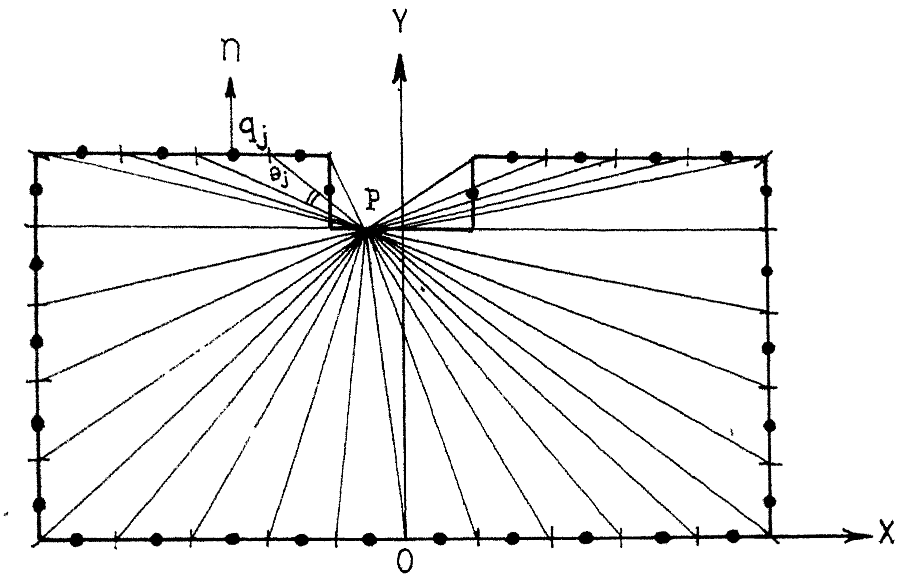

FIG. 16 - ANGLES SUBTENDED BY THE INTERVALS OF THE RECTANGULAR BOUNDARY WITH A RECTANGULAR NOTCH FOR N=32 AT ONE OF THE NODAL POINTS.

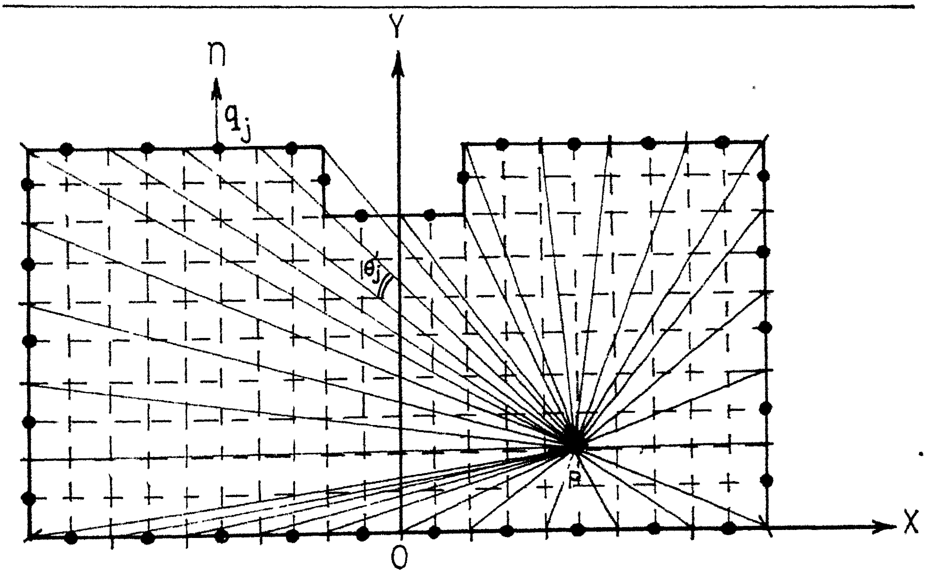

FIG. 17 - ANGLES SUBTENDED BY THE INTERVALS OF THE RECTANGULAR BOUNDARY WITH A RECTANGULAR NOTCH FOR N=32 AT ONE OF THE LATTICE POINTS WHERE CONJUGATE TORSION FUNCTION Ψ IS COMPUTED.

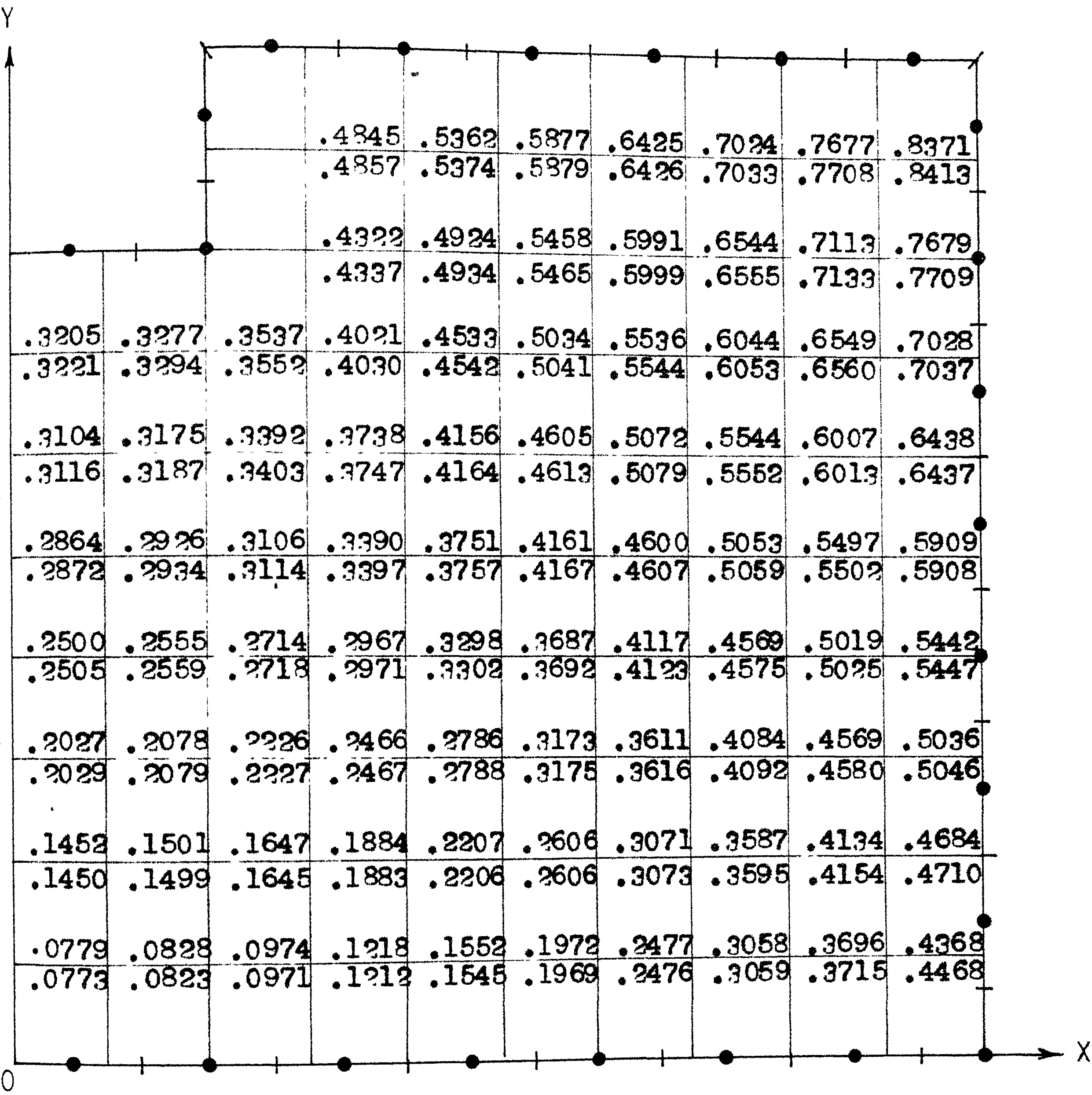

FIG. 18 - COMPUTED VALUES OF Ψ FOR N = 48 BY FIRST AND SECOND METHODS ARE SHOWN BELOW AND ABOVE THE LINE RESPECTIVELY PASSING THROUGH A GRID POINT.

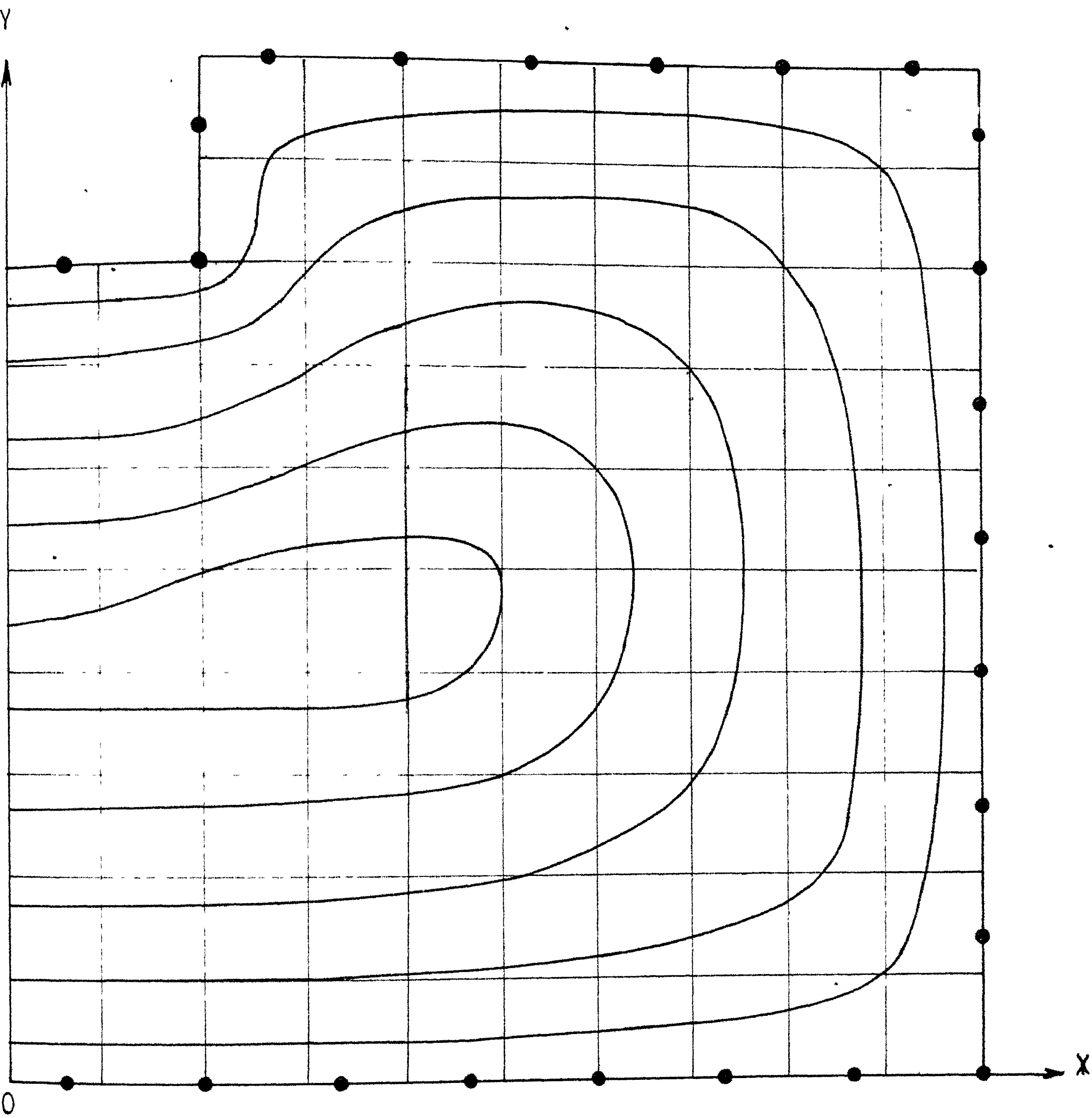

FIG. 19 - LINES OF SHEARING STRESS IN ONE HALF OF THE RECTANGULAR CROSS-SECTION (2X1) WITH A RECTANGULAR NOTCH (.4 X .2).

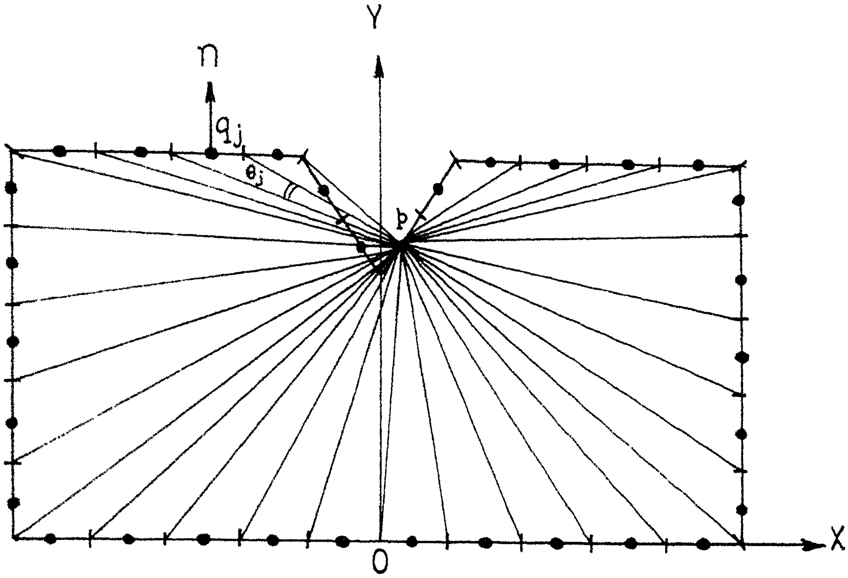

FIG. 20 - ANGLES SUBTENDED BY THE INTERVALS OF THE RECTANGULAR BOUNDARY WITH AN EQUILATERAL TRIANGULAR NOTCH AT ONE OF THE NODAL POINTS.

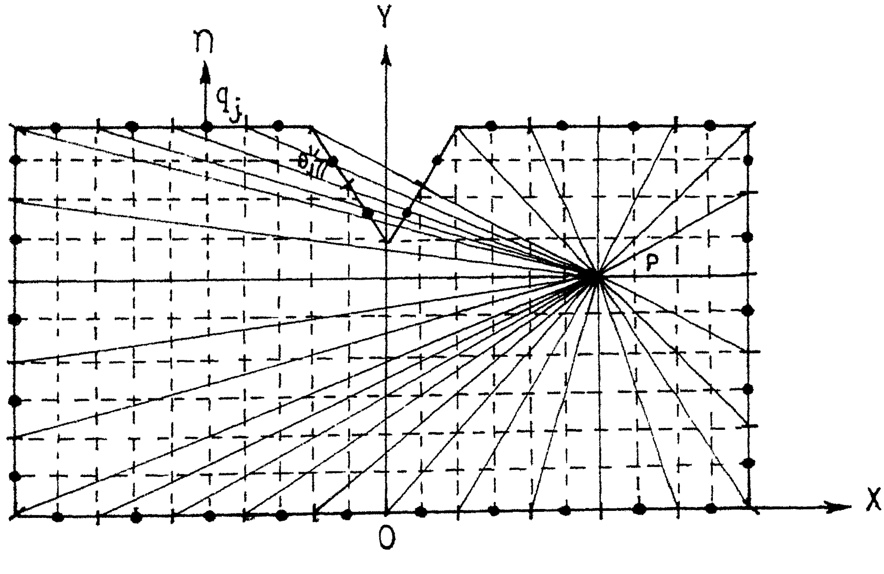

FIG. 21 - ANGLES SUBTENDED BY THE INTERVALS OF THE RECTANGULAR BOUNDARY WITH AN EQUILATERAL TRIANGULAR NOTCH AT ONE OF THE LATTICE POINTS.

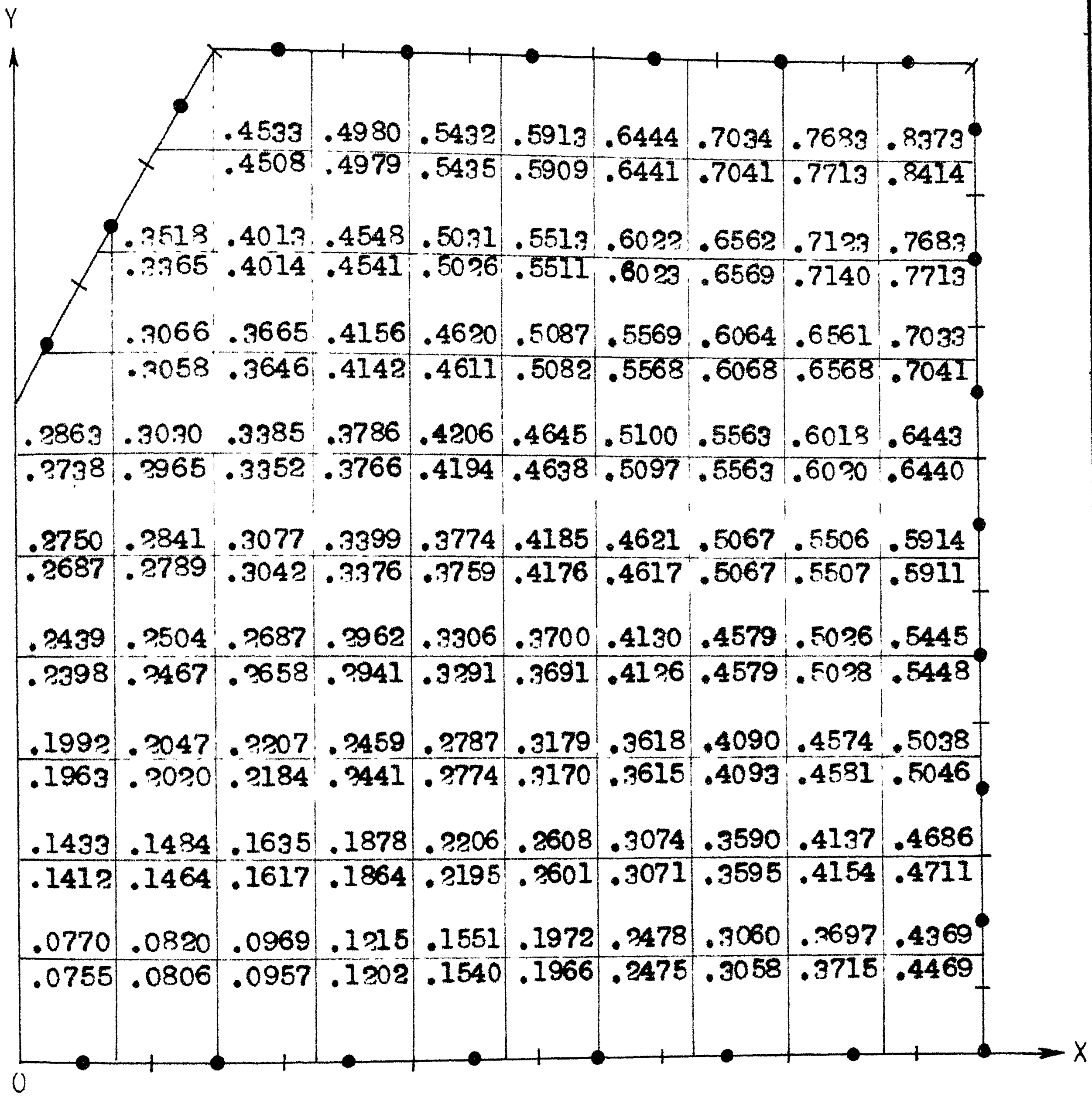

FIG. 22 - COMPUTED VALUES OF ψ FOR $N = 48$ BY FIRST AND SECOND METHODS ARE SHOWN BELOW AND ABOVE THE LINE RESPECTIVELY PASSING THROUGH A GRID POINT.

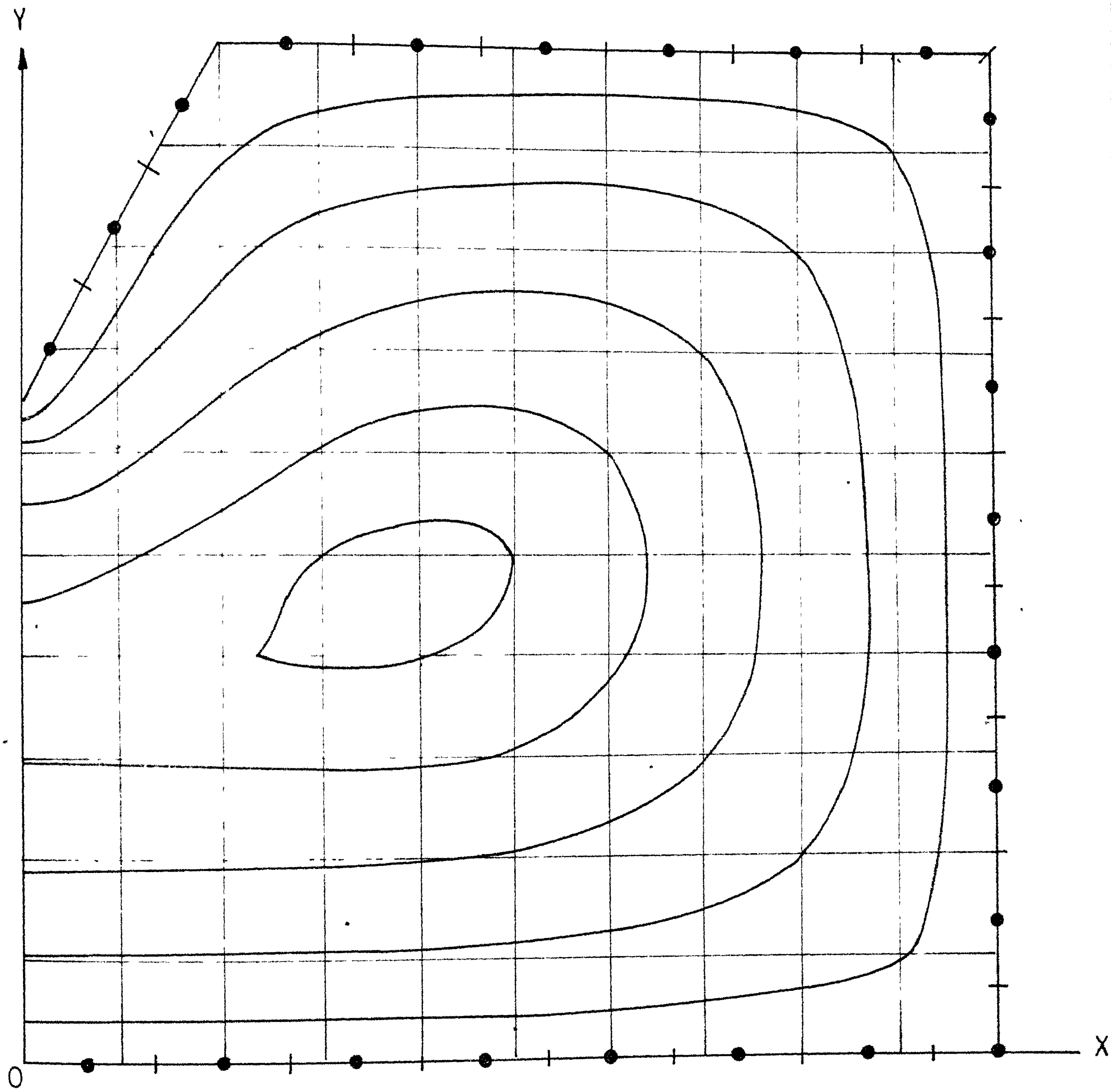

FIG. 23 - LINES OF SHEARING STRESS IN ONE HALF OF THE RECTANGULAR CROSS-SECTION (2 X 1) WITH EQUILATERAL TRIANGULAR NOTCH (.4)

TORSION PROBLEM FOR RECTANGULAR CROSS SECTION (2x1)
WITH A RECTANGULAR NOTCH (.4 x .2)

Table No. 57

Computed Values of σ for N = 48

0.2219855	0.2323921	0.2550532	0.2941085
0.3570878	0.4651543	0.5801486	2.0479730
0.7665505	0.7048410	0.6567106	0.6659880
0.7298020	0.8503882	1.6065400	1.5674010
0.7888230	0.6498188	0.5678382	0.5365738
0.7304715	0.2883319	-0.2446768	-0.0134711
-0.0134711	-0.2446769	0.2883319	0.7304715
0.5365737	0.5678383	0.6498187	0.7888231
1.5674010	1.6065400	0.8503881	0.7298019
0.6659881	0.6567105	0.7048410	0.7665505
2.0479730	0.5801487	0.4651543	0.3570877
0.2941085	0.2550532	0.2323920	0.2219855

Table No. 58

Computed Values of μ for N = 48

-0.7918845	-0.7574445	-0.6873682	-0.5793597
-0.4300358	-0.2343284	0.0183512	0.3764891
0.5716578	0.6705380	0.7632737	0.8684022
0.9912200	1.1311500	1.2812630	1.3107990
1.1862810	1.0679640	0.9642861	0.8748362
0.7853857	0.3630039	0.0639661	0.2093525
0.2093525	0.0639661	0.3630039	0.7853858
0.8748362	0.9642861	1.0679640	1.1862810
1.3107990	1.2812630	1.1311500	0.9912200
0.8684022	0.7632737	0.6705380	0.5716578
0.3764891	0.0183512	-0.2343284	-0.4300358
-0.5793597	-0.6873682	-0.7574445	-0.7918845

Table No. 59

TORSION PROBLEM FOR RECTANGULAR CROSS-SECTION (2x1) WITH A RECTANGULAR NOTCH (.4 x .2) BY FIRST METHOD

Coordinates of the point P X	Y	Computed value of ψ for N = 16	Computed value of ψ for N = 32	Computed value of ψ for N = 48	Maximum shearing stress for N = 48
.00	.10	.0743328	.0786357	.0773062	.6252285
.10	.10	.0813371	.0846562	.0823006	.6247226
.20	.10	.1055142	.0981770	.0970656	.6224781
.30	.10	.1191037	.1235789	.1212034	.6164013
.40	.10	.1513725	.1557061	.1545453	.6032401
.50	.10	.2002705	.1993738	.1968958	.5790038
.60	.10	.2721693	.2484243	.2476433	.5393934
.70	.10	.3387465	.3100994	.3058529	.4792775
.80	.10	.4247308	.3747928	.3714739	.3888136
.90	.10	.4875440	.4416454	.4468388	.3270231
.00	.20	.1436217	.1484734	.1450457	.4277475
.10	.20	.1497448	.1534113	.1499443	.4279504
.20	.20	.1661979	.1678764	.1644908	.4284933
.30	.20	.1897387	.1915979	.1882545	.4280833
.40	.20	.2236751	.2237122	.2205711	.3233743
.50	.20	.2700181	.2636350	.2605949	.4086647
.60	.20	.3266453	.3102659	.3073120	.3845648
.70	.20	.3893892	.3627092	.3594819	.3545997
.80	.20	.4522355	.4182815	.4154023	.3358024
.90	.20	.5147006	.4736649	.4710211	.3872789
.00	.30	.2007514	.2079985	.2028557	.2273595
.10	.30	.2064136	.2130472	.2078889	.2299246
.20	.30	.2227088	.2278850	.2227476	.2367013
.30	.30	.2485507	.2517141	.2467303	.2721579
.40	.30	.2837171	.2834291	.2787650	.2523036
.50	.30	.3276872	.2218075	.3175461	.2433546
.60	.30	.3784608	.3654715	.3616017	.2657735

CONTD...

Table No. 59 (CONTD.)

Coordinates of the point P X	Y	Computed value of ψ for N = 16	Computed value of ψ for N = 32	Computed value of ψ for N = 48	Maximum shearing stress for N = 48
.70	.30	.4329563	.4127152	.4091886	.2893757
.80	.30	.4876245	.4612360	.4579721	.3505068
.90	.30	.5337187	.5095369	.5045558	.4698588
.00	.40	.2462821	.2573024	.2505176	.0219520
.10	.40	.2523295	.2628238	.2559291	.0281557
.20	.40	.2699680	.2788373	.2718169	.0434966
.30	.40	.2979420	.3039467	.2971179	.0653965
.40	.40	.3346801	.3364445	.3301890	.1889134
.50	.40	.3783200	.3746648	.3591831	.1311796
.60	.40	.4266077	.4169555	.4122706	.1849849
.70	.40	.4770037	.4614166	.4575081	.2623159
.80	.40	.5273860	.5055444	.5025464	.3691650
.90	.40	.5832601	.5446784	.5446916	.5136939
.00	.50	.2795297	.2953628	.2872461	.1943485
.10	.50	.2865265	.3019583	.2933683	.1871313
.20	.50	.3068455	.3206484	.3113653	.1606940
.30	.50	.3385281	.3488112	.3397079	.1142210
.40	.50	.3786682	.3837454	.3757051	.0707956
.50	.50	.4242476	.4234119	.4167120	.0847314
.60	.50	.4726622	.4661733	.4607190	.1554925
.70	.50	.5214961	.5102610	.5059185	.2529582
.80	.50	.5677658	.5534636	.5501816	.3754391
.90	.50	.6028171	.5944819	.5908403	.5259325
.00	.60	.2985530	.3198316	.3116479	.4255590
.10	.60	.3071867	.3287080	.3187302	.4221275
.20	.60	.3321755	.3530071	.3403096	.3889384
.30	.60	.3706531	.3870468	.3746897	.2945100
.40	.60	.4173111	.4263850	.4163509	.2102401
.50	.60	.4673784	.4690382	.4612715	.1682318

Table No. 59 (CONTD.)

Coordinates of the point P X	Y	Computed value of ψ for N = 16	Computed value of ψ for N = 32	Computed value of ψ for N = 48	Maximum shearing stress for N = 48
.60	.60	.5182347	.5139658	.5079116	.1854428
.70	.60	.5681882	.5599467	.5551657	.2545679
.80	.60	.6150350	.6049409	.6013444	.3644210
.90	.60	.6461644	.6444357	.6436761	.5079799
.00	.70	.2998730	.3258628	.3221343	.6582395
.10	.70	.3113432	.3391479	.3293881	.6718505
.20	.70	.3432987	.3760956	.3552324	.7214346
.30	.70	.3948304	.4204368	.4030323	.4493410
.40	.70	.4529938	.4658426	.4541754	.3319682
.50	.70	.5098521	.5123472	.5041392	.2738923
.60	.70	.5646216	.5605661	.5543762	.2579255
.70	.70	.6176811	.6103861	.6053348	.2758252
.80	.70	.6690600	.6604328	.6559920	.3386997
.90	.70	.7189215	.7098346	.7036948	.4593979
.30	.80	.4146478	.4533136	.4336975	.6064068
.40	.80	.4911140	.5042299	.4934016	.4173090
.50	.80	.5544170	.5538350	.5464934	.3824913
.60	.80	.6124885	.6052819	.5998760	.3631796
.70	.80	.6695464	.6605299	.6555426	.3375358
.80	.80	.7215301	.7177303	.7132515	.3173600
.90	.80	.7896140	.7743766	.7709358	.3829881
.30	.90	.4351760	.4923387	.4856675	.4320864
.40	.90	.5509766	.5430944	.5374271	.4800736
.50	.90	.5997030	.5941997	.5879154	.5081177
.60	.90	.6623902	.6446784	.6425685	.4999130
.70	.90	.7236188	.7096612	.7032997	.4564969
.80	.90	.7884897	.7746208	.7708375	.3824153
.90	.90	.7240725	.8423444	.8413025	.3244678

Table No. 60

TORSION PROBLEM FOR RECTANGULAR CROSS-SECTION (2x1)
WITH A RECTANGULAR NOTCH (.4 x .2) BY SECOND METHOD

Coordinates of the point P X	Y	Computed value of ψ for N = 16	Computed value of ψ for N = 32	Computed value of ψ for N = 48	Maximum shearing stress for N = 48
.00	.10	.0852106	.0806883	.0779466	.6261755
.10	.10	.0878266	.0848940	.0827821	.6256729
.20	.10	.0967619	.1002388	.0974491	.6234166
.30	.10	.1175330	.1236441	.1218307	.6172743
.40	.10	.1647729	.1577234	.1552065	.6039856
.50	.10	.2151332	.1989440	.1971599	.5792577
.60	.10	.2478559	.2499982	.2476861	.5382827
.70	.10	.2904280	.3066405	.3058319	.4769707
.80	.10	.3814424	.3713391	.3695775	.3948645
.90	.10	.4588381	.4377560	.4368006	.3112260
.00	.20	.1463621	.1492575	.1452351	.4238480
.10	.20	.1505198	.1540778	.1501336	.4248640
.20	.20	.1637046	.1685815	.1646797	.4259307
.30	.20	.1880086	.1920837	.1884345	.4242947
.40	.20	.2242493	.2241104	.2207068	.4187663
.50	.20	.2659306	.2635623	.2606092	.4063468
.60	.20	.3084593	.3096316	.3070589	.3833059
.70	.20	.3580282	.3606900	.3586596	.3552305
.80	.20	.4186232	.4150731	.4134432	.3450223
.90	.20	.4740185	.4695297	.4684432	.3908333
.00	.30	.1914193	.2083268	.2027162	.2240340
.10	.30	.2044537	.2133394	.2077531	.2266495
.20	.30	.2195985	.2280941	.2226183	.2335615
.30	.30	.2447241	.2517730	.2465966	.2421845
.40	.30	.2785898	.2832690	.2785934	.2497160
.50	.30	.3183838	.3212921	.3172751	.2558944
.60	.30	.3620530	.3644390	.3611378	.2658444

CONTD...

Table No. 60 (CONTD.)

Coordinates of the point P X	Y	Computed value of ψ for N = 16	Computed value of ψ for N = 32	Computed value of ψ for N = 48	Maximum shearing stress for N = 48
.70	.30	.4097696	.4109958	.4084221	.2922516
.80	.30	.4610504	.4587729	.4569054	.3542297
.90	.30	.5152702	.5047425	.5035611	.4662032
.00	.40	.2424302	.2572835	.2500419	.0184890
.10	.40	.2482720	.2627538	.2554634	.0249125
.20	.40	.2653379	.2786484	.2713777	.0404622
.30	.40	.2923066	.3036061	.2967039	.0627519
.40	.40	.3271483	.3359002	.3297724	.0916421
.50	.40	.3675251	.3738339	.3687193	.1304262
.60	.40	.4114545	.4157741	.4117180	.1851591
.70	.40	.4573408	.4599282	.4568687	.2627962
.80	.40	.5029294	.5041067	.5019423	.3691087
.90	.40	.5448009	.5456709	.5441663	.5110385
.00	.50	.2736004	.2951175	.2864140	.1980260
.10	.50	.2807210	.3015921	.2925532	.1909379
.20	.50	.3010510	.3200820	.3106151	.1640363
.30	.50	.3319487	.3481201	.3390439	.1165612
.40	.50	.3702503	.3829104	.3750912	.0724346
.50	.50	.4132235	.4223418	.4161027	.0856784
.60	.50	.4587766	.4648264	.4600295	.1556742
.70	.50	.5049340	.5087433	.5052997	.2518812
.80	.50	.5485721	.5520474	.5497121	.3719374
.90	.50	.5815570	.5919747	.5909220	.5235642
.00	.60	.2901579	.3196841	.3104367	.4293410
.10	.60	.2996714	.3281218	.3175076	.4266372
.20	.60	.3261136	.3519181	.3392189	.3930919
.30	.60	.3642846	.3860445	.3738254	.2961222
.40	.60	.4088077	.4253717	.4155872	.2116197
.50	.60	.4563755	.4677936	.4605476	.1690623

Table No. 60 (CONTD.)

Coordinates of the point P X	Y	Computed value of ψ for N = 16	Computed value of ψ for N = 32	Computed value of ψ for N = 48	Maximum shearing stress for N = 48
.60	.60	.5053483	.5123717	.5079843	.1858890
.70	.60	.5544411	.5580596	.5544345	.2547066
.80	.60	.6023886	.6031633	.6007353	.3609777
.90	.60	.6481107	.6453836	.6437718	.5053038
.00	.70	.2869363	.3269973	.3205458	.6521835
.10	.70	.2011043	.3385346	.3276789	.6657989
.20	.70	.3394382	.3734140	.3537088	.7170865
.30	.70	.3907176	.4194653	.4021263	.4536467
.40	.70	.4446205	.4648877	.4533045	.3332159
.50	.70	.4981369	.5110003	.5033510	.2739131
.60	.70	.5517879	.5587083	.5535664	.2583290
.70	.70	.6051464	.6078376	.6043830	.2782262
.80	.70	.6584236	.6571509	.6549141	.3411478
.90	.70	.7179440	.7041428	.7028179	.4565335
.30	.80	.4161931	.4542073	.4321921	.6033923
.40	.80	.4813734	.5035018	.4924142	.4204238
.50	.80	.5393746	.5524443	.5457656	.3799176
.60	.80	.5985489	.6034540	.5991263	.3600201
.70	.80	.6568560	.6572273	.6544046	.3398522
.80	.80	.7106707	.7131727	.7113242	.3312311
.90	.80	.7639784	.7691286	.7678857	.3791347
.30	.90	.4591910	.4916140	.4845189	.4224207
.40	.90	.5254215	.5425970	.5362132	.4763431
.50	.90	.5769843	.5914100	.5876647	.5047965
.60	.90	.6459222	.6452947	.6424643	.4963410
.70	.90	.7149526	.7041355	.7023586	.4530376
.80	.90	.7633160	.7690326	.7677489	.3783612
.90	.90	.8050744	.8371816	.8370772	.3027644

TORSION PROBLEM FOR RECTANGULAR CROSS-SECTION (2x1)
WITH AN EQUILATERAL TRIANGULAR NOTCH (.4)

Table No. 61

Computed Values of σ for N = 48

0.2258570	0.2356970	0.2575795	0.2960070
0.3586570	0.4666915	0.5817639	2.0524910
0.7682365	0.7063550	0.6579592	0.6670147
0.7307338	0.8514648	1.6087690	1.5690100
0.7885911	0.6479278	0.5623400	0.5241188
0.6690443	0.3573506	-0.0342750	-0.2024533
-0.2024533	-0.0342750	0.3573506	0.6690444
0.5241188	0.5623400	0.6479276	0.7885913
1.5690090	1.6087700	0.8514648	0.7307336
0.6670149	0.6579592	0.7063548	0.7682367
2.0524910	0.5817640	0.4666915	0.3586569
0.2960071	0.2575795	0.2356970	0.2258570

Table No. 62

Computed Values of μ for N = 48

-0.7886278	-0.7547061	-0.6854231	-0.5781936
-0.4294688	-0.2341631	0.0182302	0.3759756
0.5705275	0.6688912	0.7611632	0.8659500
0.9886560	1.1287750	1.2793050	1.3085970
1.1830740	1.0633060	0.9576579	0.8662840
0.7788006	0.4782666	0.1977915	-0.0917742
-0.0917742	0.1977915	0.4782666	0.7788007
0.8662840	0.9576579	1.0633060	1.1830740
1.3085970	1.2793305	1.1287750	0.9886560
0.8659500	0.7611632	0.6688912	0.5705275
0.3759756	0.0182302	-0.2341631	-0.4294688
-0.5781936	-0.6854231	-0.7547061	-0.7886278

Table No. 63

TORSION PROBLEM FOR RECTANGULAR CROSS-SECTION (2x1) WITH A EQUILATERAL TRIANGULAR NOTCH (.4) BY FIRST METHOD

Coordinates of the point P X	Y	Computed value of ψ for N = 16	Computed value of ψ for N = 32	Computed value of ψ for N = 48	Maximum shearing stress for N = 48
.00	.10	.0792935	.0756644	.0755064	.6059605
.10	.10	.0860940	.0818454	.0806251	.6069623
.20	.10	.1096309	.0956621	.0957083	.6085000
.30	.10	.1229328	.1215349	.1202412	.6069201
.40	.10	.1546933	.1540956	.1539501	.5976829
.50	.10	.2029707	.1981954	.1965800	.5762337
.60	.10	.2741516	.2476722	.2475071	.5383007
.70	.10	.3402327	.3095408	.3058141	.4790406
.80	.10	.4257848	.3744491	.3714789	.3889070
.90	.10	.4881558	.4414915	.4468669	.3271594
.00	.20	.1533986	.1423227	.1411921	.4039000
.10	.20	.1592352	.1475531	.1463923	.4067374
.20	.20	.1749287	.1627761	.1616994	.4134000
.30	.20	.1974527	.1874784	.1863713	.4192918
.40	.20	.2301976	.2205650	.2194805	.4185827
.50	.20	.2752647	.2613320	.2600665	.4070126
.60	.20	.3306497	.3086429	.3071205	.3841638
.70	.20	.3912832	.3616224	.3594579	.3546916
.80	.20	.4541481	.4176196	.4154384	.3360057
.90	.20	.5157305	.4733538	.4710548	.3875108
.00	.30	.2159351	.1980268	.1962864	.1928555
.10	.30	.2211593	.2036658	.2019710	.2016725
.20	.30	.2362454	.2199729	.2183883	.2208061
.30	.30	.2603514	.2455578	.2440702	.2388451
.40	.30	.2935184	.2788818	.2774289	.2507608
.50	.30	.3354543	.3185599	.3170387	.2577372
.60	.30	.3843247	.3632185	.3615244	.2662179
.70	.30	.4371283	.4112167	.4092845	.2898671

CONTD...

Table No. 63 (CONTD.)

Coordinates of the point P		Computed value of ψ for N = 16	Computed value of ψ for N = 32	Computed value of ψ for N = 48	Maximum shearing stress for N = 48
X	Y				
.80	.30	.4903095	.4603247	.4580958	.3510686
.90	.30	.5350701	.5091156	.5046330	.4705560
.00	.40	.3677882	.2422013	.2397632	.0378385
.10	.40	.2731272	.2489901	.2466720	.0338578
.20	.40	.2888339	.2678671	.2658107	.0473099
.30	.40	.3140876	.2959514	.2941307	.0698779
.40	.40	.3477840	.3308330	.3291373	.0958791
.50	.40	.3884622	.3707864	.3690990	.1321646
.60	.40	.4341040	.4143125	.4125582	.1858068
.70	.40	.4822393	.4596723	.4578620	.2633879
.80	.40	.5306976	.5044843	.5028273	.3704734
.90	.40	.5848517	.5441689	.5448422	.5151462
.00	.50	.3088594	.2721375	.2687187	.3[illegible]99910
.10	.50	.3146307	.2820194	.2789244	.2625453
.20	.50	.3318449	.3066911	.3042104	.1794661
.30	.50	.3593248	.3396752	.3376441	.1055657
.40	.50	.3949643	.3777474	.3759328	.0488506
.50	.50	.4364200	.4193978	.4176259	.0762204
.60	.50	.4813860	.4634690	.4616544	.1555209
.70	.50	.5274471	.5084855	.5066559	.2550345
.80	.50	.5714819	.5523794	.5506703	.3778968
.90	.50	.6046635	.5939744	.5910807	.5283708
.00	.60	.3386389	.2788813	.2737650	.5412708
.10	.60	.3443828	.3006533	.2965002	.5094150
.20	.60	.3644438	.3378460	.3351856	.3593311
.30	.60	.3963741	.3785672	.3765708	.2485807
.40	.60	.4363484	.4211234	.4193672	.1776277
.50	.60	.4808557	.4655271	.4637596	.1546736

Table No. 63 (CONTD.)

Coordinates of the point P X	Y	Computed value of ψ for N = 16	Computed value of ψ for N = 32	Computed value of ψ for N = 48	Maximum shearing stress for N = 48
.70	.60	.5743121	.5583438	.5563202	.2581316
.80	.60	.6187965	.6039547	.6020340	.3682440
.90	.60	.6580581	.6439579	.6439973	.5113792
.10	.70	.3588943	.3081570	.3058412	.4951000
.20	.70	.3850318	.3667554	.3645522	.5030234
.30	.70	.4254311	.4157523	.4142051	.3618047
.40	.70	.4734482	.4625942	.4611422	.2928670
.50	.70	.5232418	.5099035	.5082497	.2643059
.60	.70	.5733069	.5587658	.5568032	.2599844
.70	.70	.6232205	.6091317	.6067587	.2810156
.80	.70	.6724430	.6596449	.6567856	.3435116
.90	.70	.7205712	.7094632	.7040512	.4631987
.10	.80	.3563497	.3436910	.3365397	.8315000
.20	.80	.3905094	.4031601	.4013772	.5353131
.30	.80	.4469608	.4544719	.4541426	.4332495
.40	.80	.5098943	.5033932	.5025025	.3974066
.50	.80	.5653449	.5525768	.5510768	.3865895
.60	.80	.6190895	.6041761	.6023242	.3702912
.70	.80	.6736061	.6597055	.6568838	.3437990
.80	.80	.7239178	.7171980	.7139666	.3221783
.90	.80	.7907193	.774154(	.7712511	.3867640
.20	.90	.3742307	.4444508	.4507945	.3210000
.30	.90	.4596344	.4965386	.4979364	.4747400
.40	.90	.5618434	.5431652	.5434858	.5166151
.50	.90	.6051016	.5936657	.5909320	.5303934
.60	.90	.6651688	.6441867	.6441177	.5127538
.70	.90	.7254160	.7092701	.7041188	.4638913
.80	.90	.7896204	.7743565	.7712657	.3868630
.90	.90	.7243543	.8421830	.8414150	.3270962

Table No. 64

TORSION PROBLEM FOR RECTANGULAR CROSS-SECTION (2x1) WITH A EQUILATERAL TRIANGULAR NOTCH (.4) BY SECOND METHOD

Coordinates of the point P X	Y	Computed value of ψ for N = 16	Computed value of ψ for N = 32	Computed value of ψ for N = 48	Maximum shearing stress for N = 48
.00	.10	.0908684	.0790160	.0770485	.6163170
.10	.10	.0932429	.0833059	.0819727	.6168937
.20	.10	.1015299	.0988293	.0968623	.6173297
.30	.10	.1215454	.1224968	.1215131	.6142903
.40	.10	.1685870	.1568358	.1551259	.6035208
.50	.10	.2185900	.1982935	.1972368	.5803613
.60	.10	.2501202	.2495411	.2478356	.5400363
.70	.10	.2915646	.3063343	.3059899	.4787412
.80	.10	.3824853	.3711533	.3697019	.3963752
.90	.10	.4596349	.4376675	.4368668	.3121739
.00	.20	.1572597	.1458077	.1432634	.4107250
.10	.20	.1611021	.1507936	.1483786	.4136662
.20	.20	.1734272	.1657284	.1634651	.4191879
.30	.20	.1965639	.1897918	.1878458	.4220438
.40	.20	.2315684	.2223757	.2206284	.4194952
.50	.20	.2719046	.2623071	.2608431	.4084795
.60	.20	.3129415	.3087576	.3074201	.3857927
.70	.20	.3611349	.3601093	.3590154	.3574835
.80	.20	.4206666	.4147204	.4137138	.3469291
.90	.20	.4750380	.4693637	.4685872	.3924816
.00	.30	.2162168	.2027408	.1991935	.2031570
.10	.30	.2207737	.2080825	.2047057	.2103249
.20	.30	.2345941	.2236677	.2207002	.2264253
.30	.30	.2578225	.2483529	.2458742	.2421854
.40	.30	.2895223	.2807735	.2787322	.2529250
.50	.30	.3270947	.3195360	.3178579	.2597791
.60	.30	.3686295	.3632378	.3618423	.2694051
.70	.30	.4144248	.4102056	.4090524	.2952985

CONTD...

Table No. 64 (CONTD.)

Coordinates of the point P		Computed value of ψ for N = 16	Computed value of ψ for N = 32	Computed value of ψ for N = 48	Maximum shearing stress for N = 48
X	Y				
.80	.30	.4640508	.4582948	.4573610	.3568193
.90	.30	.5168720	.5045169	.5037981	.4686160
.00	.40	.2661218	.2488394	.2438948	.0207745
.10	.40	.2712068	.2549922	.2504166	.0243909
.20	.40	.2861850	.2724904	.2687351	.0454248
.30	.40	.3101950	.2991691	.2962429	.0707665
.40	.40	.3417375	.3328465	.3305742	.0984002
.50	.40	.3788881	.3717668	.3699758	.1356483
.60	.40	.4198989	.4143899	.4129548	.1897132
.70	.40	.4632556	.4590259	.4578672	.2667368
.80	.40	.5066683	.5035618	.5026226	.3727461
.90	.40	.5466024	.5454111	.5445085	.5145074
.00	.50	.3057660	.2822578	.2750386	.2879350
.10	.50	.3116516	.2903416	.2840989	.2451406
.20	.50	.3286314	.3121615	.3076709	.1704702
.30	.50	.3549442	.3430760	.3399075	.1006542
.40	.50	.3883902	.3797153	.3773881	.0501621
.50	.50	.4269062	.4202658	.4184874	.0814295
.60	.50	.4686803	.4634551	.4620563	.1594769
.70	.50	.5117402	.5078494	.5067332	.2572516
.80	.50	.5528103	.5515051	.5506308	.3768416
.90	.50	.5834526	.5917170	.5913689	.5281550
.00	.60	.3333975	.2984694	.2863078	.7129424
.10	.60	.3408086	.3115780	.3030438	.5134582
.20	.60	.3616745	.3430866	.3385276	.3536138
.30	.60	.3925979	.3815382	.3786126	.2476472
.40	.60	.4300294	.4227472	.4206428	.1791235
.50	.60	.4716540	.4660891	.4644846	.1580316
[illegible]	[illegible]	[illegible]	.5112968	.5099569	.1890622

Table No. 64 (CONTD.)

Coordinates of the point P X	Y	Computed value of ψ for N = 16	Computed value of ψ for N = 32	Computed value of ψ for N = 48	Maximum shearing stress for N = 48
.70	.60	.5615908	.5572932	.5562688	.2611681
.80	.60	.6068447	.6026882	.6018431	.3671606
.90	.60	.6504332	.6451518	.6442947	.5107924
.10	.70	.3561264	.3139794	.3065927	.4551300
.20	.70	.3846490	.3688696	.3665434	.5176248
.30	.70	.4240193	.4176747	.4155895	.3652221
.40	.70	.4676296	.4636376	.4620479	.2958143
.50	.70	.5136302	.5100185	.5087423	.2669551
.60	.70	.5620815	.5579434	.5568839	.2634015
.70	.70	.6118324	.6072798	.6064006	.2860162
.80	.70	.6625531	.6567882	.6560704	.3481422
.90	.70	.7203195	.7039635	.7033477	.4622999
.10	.80	.3523544	.2916218	.3518324	.9950000
.20	.80	.3969099	.4050756	.4013469	.4829366
.30	.80	.4522963	.4558117	.4547775	.4404195
.40	.80	.5031384	.5038349	.5029605	.4027083
.50	.80	.5527552	.5521996	.5513439	.3874213
.60	.80	.6069236	.6030745	.6022488	.3703380
.70	.80	.6620683	.6568914	.6561880	.3487162
.80	.80	.7137615	.7129358	.7123058	.3380619
.90	.80	.7654376	.7690045	.7683250	.3842249
.20	.90	.3987172	.4452977	.4532834	.3080000
.30	.90	.4894709	.4965974	.4979552	.4731992
.40	.90	.5399540	.5435601	.5432284	.5190837
.50	.90	.5851072	.5915115	.5912613	.5317509
.60	.90	.6506513	.6451404	.6443835	.5127481
.70	.90	.7176783	.7039810	.7034122	.4629644
.80	.90	.7649072	.7689092	.7683151	.3843343
.90	.90	.8055338	.8371203	.8373280	.3063190

APPENDIX I

Fortran programme for Dirichlet Problem for circular domain y First Method when $\Phi(p) = x$, using trapezoidal rule.

```
C PROGRAMME DIRICHLET(SAXENA,R.S.)
      DIMENSION A(32,32),R(32,32),S(32,32),U(32,32),V(32,32)
      DIMENSION X(32),Y(32),Z(32),UL(32),UR(32),H(32),SX(32),XO(8),YO(8)
      DOUBLE PRECISION A,R,S,U,V,X,Y,Z,UL,UR,SX,F1,F2,F3,ST,FS,FZ,FX,FY
      READ 70, (XO(J),J=1,8)
      READ 70, (YO(J),J=1,8)
      PRINT 100
      PRINT 200
      PRINT 300
      DO 700 INI=1,4
      READ 150, N
      PRINT 250, N
      READ 60, (H(J),J=1,N)
      PI=3.1415926535897930
      RADIUS=1./PI
      I=0
  135 I=I+1
      IF(I-N) 145,145,155
  145 CONTINUE
      AI=I-1
      AN=N
      SX(I)=2.*PI*AI/AN
      Y(I)=-AN*DCOS(SX(I))/(2.*PI)
      DO 51 J=1,N
      AJ=J-1
      IF (I-J) 115,125,115
  115 FX=PI*AJ/AN-SX(I)/2.
      FY=(2.*RADIUS*DSIN(FX))**2
      FZ=DLOG(FY)/2.
```

```
      GO TO 51
125   SN=H(J)/2
      F1=SN**3/(72.*(RADIUS**2))
      F2=SN**5/(14400.*(RADIUS**4))
      F3=SN**7/(1270080.*(RADIUS**6))
      ST=(DLOG(SN)-1.)*SN
      FS=ST-(F1+F2+F3)
      FZ=FS/SN
51    A(I,J)=FZ
      GO TO 135
155   K=1
45    KL=K
      KR=K+1
      DO 32 I=1,N
      DO 32 J=1,N
      U(I,J)=0
32    V(I,J)=0
41    KL=K
      DO 10 I=KL,N
10    S(I,K)=A(I,K)-U(I,K)
      IF (K-N) 15,16,16
15    DO 11 J=KR,N
      R(K,J)=A(K,J)/S(K,K)-V(K,J)
11    R(K,K)=1.
      K=K+1
      DO 12 I=K,N
      DO 12 J=1,KL
12    U(I,K)=S(I,J)*R(J,K)+U(I,K)
      IF (K-N) 21,14,14
21    KR=K+1
      S(K,K)=A(K,K)-U(K,K)
      DO 13 I=KR,N
      DO 13 J=1,KL
13    V(K,I)=V(K,I)+S(K,J)*R(J,I)/S(K,K)
```

```
14   GO TO 41
16   R(M,M)=1.
     K=1
     DO 17 I=1,N
     UL(I)=0.
17   UR(I)=0.
18   Z(K)=Y(K)/S(K,K)-UL(K)
     IF (K-N) 20,19,19
20   DO 22   I=1,K
22   UL(K+1)=UL(K+1)+S(K+1,I)*Z(I)/S(K+1,K+1)
     K=K+1
     GO TO 18
19   M=N
25   KM=M
     X(KM)=Z(KM)-UR(KM)
     IF (KM-1) 26,26,27
27   DO 28 I=KM,N
28   UR(KM-1)=UR(KM-1)+R(KM-1,I)*X(I)
     M=M-1
     GO TO 25
26   PRINT 40, (X(I),I=1,N)
     PRINT 778
     DO 777 I=1,8
     RV=DSQRT(XO(I)**2+YO(I)**2)
     AI=I
     SI=PI*(AI-1.)/4.
     AV=XO(I)
     CV=0.
     DO 151 K=1,N
     AN=N
     PK=K-1
     B1=(2.*PI*PK/AN)-SI
     B2=COSF(B1)
     B3=1./(PI**2)+RV**2-2.*RV*B2/PI
```

```
      B4=DLOG(B3)
 151  CV=CV-(B4*X(K)/AN)
      ERROR=DABS(AV-CV)
 777  PRINT 30,XO(I),YO(I),AV,CV,ERROR
 700  CONTINUE
 60   FORMAT (4D20.16)
 40   FORMAT (X,*SIGMAS-*/(4D20.16))
 100  FORMAT (X,* COMPUTED VALUES OF FI AT P(XO,YO) BY TRAP. RULE*)
 200  FORMAT (X,* FUNCTION-*,X,*U=X*)
 300  FORMAT (X,* CURVE-CIRCLE OF RADIUS 1./PI*)
 778  FORMAT (7X,*XO*,14X,*YO*,12X,*A.V.*,12X,*C.V.*,12X,*ERROR*)
 150  FORMAT (I3)
```

Fortran programme for Dirichlet Problem for circular domain by First Method when $\Phi(p)=x$, using Gauss Legendre quadrature formula.

```
C  C  PROGRAMME DIRICHLET(SAXENA,R.S.)
      DIMENSION  A(32,32),R(32,32),S(32,32),U(32,32),V(32,32)
      DIMENSION X(32),Y(32),Z(32),UL(32),UR(32),SI(32),H(32),XO(8),YO(8)
      DOUBLE PRECISION A,R,S,U,V,X,Y,Z,UL,UR,XMOD,B1,B2,B3,B4,B5,B
      CALL FLUN(1000)
      READ 70, (XO(J),J=1,8)
      READ 70, (YO(J),J=1,8)
      PRINT 100
      PRINT 200
      PRINT 300
      DO 700 INI=1,4
      READ 150,N
      PRINT 250, N
      READ 60, (SI(J),J=1,N)
      READ 60, (H(J),J=1,N)
      PI=3.141592653589793O
      RADIUS=1./PI
      I=0
```

```
5     I=I+1
      IF (I-N) 55,55,65
55    CONTINUE
      FI=PI*SI(I)
      Y(I)=- DCOS(FI)/PI
      DO 31 J=1,N
      TI=SI(I)+1.
      XN=SI(J)+1.-TI
      XMOD=2.*RADIUS DSIN(XN/(2.*RADIUS))
      IF (XMOD) 33,35,34
33    XMOD=-XMOD
34    B=DLOG(XMOD)
      GO TO 31
35    ZP=H(J)/2.
      B1=ZP**3/(72.*(RADIUS**2))
      B2=ZP**5/(14400.*(RADIUS**4))
      B3=ZP**7/(1270080.* (RADIUS**6))
      B4=(DLOG(ZP)-1.)*ZP
      B5=B4-(B1+B2+B3)
      B=B5/ZP
31    A(I,J)=H(J)*B
      GO TO 5
65    K=1
45    KL=K
      KR=K+1
      DO 32 I=1,N
      DO 32 J=1,N
      U(I,J)=0.
32    V(I,J)=0.
41    KL=K
      DO 10 I=KL,N
10    S(I,K)=A(I,K)-U(I,K)
      IF (K-N) 15,16,16
15    DO 11 J=KR,N
```

```
      R(K,J)=A(K,J)/S(K,K)-V(K,J)
11    R(K,K)=1.
      K=K+1
      DO 12 I=K,N
      DO 12 J=1,KL
12    U(I,K)=S(I,J)*R(J,K)+U(I,K)
      IF (K-N) 21,14,14
21    KR=K+1
      S(K,K)=A(K,K)-U(K,K)
      DO 13 I=KR,N
      DO 13 J=1,KL
13    V(K,I)=V(K,I)+S(K,J)*R(J,I)/S(K,K)
14    GO TO 41
16    R(N,N)=1.
      K=1
      DO 17 I=1,N
      UL(I)=0.
17    UR(I)=0.
18    Z(K)=Y(K)/S(K,K)-UL(K)
      IF (K-N) 20,19,19
20    DO 22  I=1,K
22    UL(K+1)=UL(K+1)+S(K+1,I)*Z(I)/S(K+1,K+1)
      K=K+1
      GO TO 18
19    M=N
25    KM=M
      X(KM)=Z(KM)-UR(KM)
      IF(KM-1) 26,26,27
27    DO 28 I=KM,N
28    UR(KM-1)=UR(KM-1)+R(KM-1,I) X(I)
      M=M-1
      GO TO 25
26    PRINT 40,(X(I),I=1,N)
      PRINT 778
```

```
      DO 75 I=1,8
      AV=XO(I)
      CV=0.
      DO 51 J=1,N
      XI=RADIUS*DCOS(SI(J)/RADIUS)
      YI=RADIUS*DSIN(SI(J)/RADIUS)
      FX=(XI-XO(I))**2+(YI-YO(I))**2
      YM=-DLOG(FX)/2.
   51 CV=CV+YM*H(J)*X(J)
      ERROR=DABS(CV-AV)
   75 PRINT 30, XO(I),YO(I),AV,CV,ERROR
  700 CONTINUE
   40 FORMAT (X,*SIGMAS-*/(4D20.16))
   30 FORMAT (5D16.8)
  100 FORMAT (X,*COMPUTED VALUES OF FI AT P(XO,YO) BY GAUSS QUAD.*)
  200 FORMAT (X,*FUNCTION-*,X,*U=X*)
  300 FORMAT (X,*CURVE-CIRCLE OF RADIUS 1./PI*)
  778 FORMAT (7X,*XO*,14X,*YO,*,12X,*A.V.*,12X,*C.V.*,12X,*ERROR*)
  150 FORMAT (I3)
  250 FORMAT (X,2HN=,I3)
   60 FORMAT (4D20.16)
   70 FORMAT (8D7.5)
      STOP
```

APPENDIX II

Fortran programme for Dirichlet Problem for circular domain y Second Method when $g(p) = x^2 - y^2$, using trapezoidal rule.

```
C   C PROGRAMME DIRICHLET, (SAXENA,R.S.)
      DIMENSION A(32,32),R(32,32),S(32,32),U(32,32),V(32,32)
      DIMENSION X(32),Y(32),Z(32),UL(32),UR(32),H(32),SX(32),XO(8),YO(8)
      DOUBLE PRECISION A,R,S,U,V,X,Y,Z,UL,UR,SX,F1,F2,F3,ST,FS,FZ,FX,FY
      READ 70, (XO(J),J=1,8)
      READ 70, (YO(J),J=1,8)
      PRINT 100
      DO 700 INI=1,4
      READ 80, N
      PRINT 90, N
      PI=3.1415926535897930
      RADIUS=1./PI
      I=0
 135  I=I+1
      IF(I-N) 145,145,155
 145  CONTINUE
      AI=I-1
      AN=N
      SX(I)=2.*PI*AI/AN
      Y(I)=2.*AN*(RADIUS**2)*DCOS(2.*SX(I))
      DO 51 J=1,N
      IF (I-J) 150,125,150
 125  A(I,J)=AN+1
      GO TO 51
 150  A(I,J)=1.
 51   CONTINUE
      GO TO 135
 155  K=1
```

```
45   KL=K
     KR=K+1
     DO 32 I=1,N
     DO 32 J=1,N
     U(I,J)=0
32   V(I,J)=0
41   KL=K
     DO 10 I=KL,N
10   S(I,K)=A(I,K)-U(I,K)
     IF (K-N) 15,16,16
15   DO 11 J=KR,N
     R(K,J)=A(K,J)/S(K,K)-V(K,J)
11   R(K,K)=1.
     K=K+1
     DO 12 I=K,N
     DO 12 J=1,KL
12   U(I,K)=S(I,J) * R(J,K)+U(I,K)
     IF (K-N) 21,14,14
21   KR=K+1
     S(K,K)=A(K,K)-U(K,K)
     DO 13 I=KR,N
     DO 13 J=1,KL
13   V(K,I)=V(K,I)+S(K,J) * R(J,I)/S(K,K)
14   GO TO 41
16   R(N,N)=1.
     K=1
     DO 17 I=1,N
     UL(I)=0.
17   UR(I)=0.
18   Z(K)=Y(K)/S(K,K)-UL(K)
     IF (K-N) 20,19,19
20   DO 22 I=1,K
22   UL(K+1)=UL(K+1)+S(K+1,I) * Z(I)/S(K+1,K+1)
     K=K+L
```

```
      GO TO 19
19    M=N
25    KM=M
      X(KM)=Z(KM)-UR(KM)
      IF (KM-1) 26,26,27
27    DO 28 I=KM,N
28    UR(KM-1)=UR(KM-1)+R(KM-1,I)*X(I)
      M=M-1
      GO TO 25
26    PRINT 40, (X(I),I=1,N)
      PRINT 778
      DO 777 I=1,8
      RV=DSQRT(XO(I)**2+YO(I)**2)
      AI=I
      SI=PI *(AI-1.)/4.
      AV=XO(I)**2-YO(I)**2
      CV=0.
      DO 151 K=1,N
      AN=N
      PK=K-1
      B1=(2.* PI * PK/AN)-SI
      B2=DCOS(B1)
      B3=RADIUS -RV* B2
      B4=RADIUS**2+RV**2-2.*RADIUS*RV*B2
      B5=RADIUS*B3/(AN*B4)
151   CV=CV+B5*X(K)
      ERROR=DABS(AV-CV)
777   PRINT 30,XO(I),YO(I),AV,CV,ERROR
700   CONTINUE
80    FORMAT (I3)
90    FORMAT (X,2HN=,I3)
40    FORMAT (X,*SIGMAS*/(4D20.16))
70    FORMAT (8D7.5)
30    FORMAT (5D16.8)
```

```
778   FORMAT (7X,*XO*,14X,*YO*,12X,*A.V.*,12X,*C.V.*,12X,*ERROR*)
100   FORMAT(X,*COMPUTED VALUES OF FI AT P(XO,YO) FOR TRAPEZOIDAL RULE*)
      STOP
      END
```

Fortran programme for Dirichlet Problem for circular domain Second Method when $g(p) = x^2 - y^2$, using Gauss Legendre quadrature rmula

```
 C    PROGRAMME DIRICHLET (SAXENA,R.S.)
      DIMENSION A(32,32),R(32,32),S(32,32),U(32,32),V(32,32)
      DIMENSION X(32),Y(32),Z(32),UL(32),UR(32),SI(32),H(32),XO(8),YO(8)
      DOUBLE PRECISION A,R,S,U,V,X,Y,Z,UL,UR,XMOD,B1,B2,B3,B4,B5,B
      CALL FLUN (1000)
      READ 70, (XO(J),J=1,8)
      READ 70, (YO(J),J=1,8)
      PRINT 100
      PRINT 200
      PRINT 300
      DO 700 INI=1,4
      READ 80, N
      READ 60, (SI(J),J=1,N)
      READ 60, (H(J),J=1,N)
      PRINT 90, N
      PI=3.1415926535897930
      RADIUS=1./PI
      I=0
 5    I=I+1
      IF (I-N) 55,55,65
 55   CONTINUE
      FI=PI*SI(I)
      Y(I)=4.*(RADIUS**2)*DCOS(2.*FI)
      DO 31 J=1,N
```

```
      IF (I-J)= 131, 135,131
135   A(I,J)=H(J)+2.
      GO TO 31
131   A(I,J)=H(J)
31    CONTINUE
      GO TO 5
65    K=1
45    KL=K
      KR=K+1
      DO 32 I=1,N
      DO 32 J=1,N
      U(I,J)=0.
32    V(I,J)=0.
41    KL=K
      DO 10 I=KL,N
10    S(I,K)=A(I,K)-U(I,K)
      IF(K-N) 15,16,16
15    DO 11 J=KR,N
      R(K,J)=A(K,J)/S(K,K)-V(K,J)
11    R(K,K)=1.
      K=K+1
      DO 12 I=K,N
      DO 12 J=1,KL
12    U(I,K)=S(I,J)*R(J,K)+U(I,K)
      IF (K-N) 21,14,14
21    KR=K+1
      S(K,K)=A(K,K)-U(K,K)
      DO 13 I=KR,N
      DO 13 J=1,KL
13    V(K,I)=V(K,I)+S(K,J)*R(J,I)/S(K,K)
14    GO TO 41
16    R(N,N)=1.
      K=1
      DO 17 I=1,N
```

```
      UL(I)=0.
17    UR(I)=0.
18    Z(K)=Y(K)/S(K,K)-UL(K)
      IF (K-N) 20,19,19
20    DO 22 I=1,K
22    UL(K+1)=UL(K+1)+S(K+1,I)*Z(I)/S(K+1,K+1)
      K=K+1
      GO TO 18
19    M=N
25    KM=M
      X(KM)=Z(KM)-UR(KM)
      IF (KM-1) 26,26,27
27    DO 28 I=KM,N
28    UR(KM-1)=UR(KM-1)+R(KM-1,I)*X(I)
      M=M-1
      GO TO 25
26    PRINT 40, (X(I),I=1,N)
      PRINT 778
      DO 75 I=1,8
      RV=DSQRT(XO(I)**2+YO(I)**2)
      AI=1
      CI=PI*(AI-1.)/4.
200   FORMAT (X,* FUNCTION-*,X,*U=X**2-Y**2*)
      CV=0.
      DO 51 J=1,N
      B1=PI*SI(J)-CI
      B2=RADIUS+RV*DCOS(B1)
      B3=RADIUS**2+RV**2+2.*RADIUS*RV*DCOS(B1)
      B4= RADIUS*H(J)*B2/(2.*B3)
51    CV=CV+X(J)*B4
      ERROR=DABS (CV-AV)
75    PRINT 30, XO(I),YO(I), AV,CV,ERROR
700   CONTINUE
      AV=XO(I)**2-YO(I)**2
```

```
300 FORMAT (X,*CURVE-CIRCLE OF RADIUS 1./PI*)
40  FORMAT (X,* SIGMAS */(4D20.16))
60  FORMAT (4D20.16)
30  FORMAT (5D16.8)
70  FORMAT (8D7.5)
80  FORMAT (I3)
90  FORMAT (X,2HN=,I3)
778 FORMAT (7X,*XO*,14X,*YO*,12X,*A.V.*,12X,*C.V.*,12X,*ERROR*)
100 FORMAT(X,* COMPUTED VALUES OF FI AT P(XO,YO) FOR GAUSS QUAD.*)
    STOP
    END
```

APPENDIX III

Autocode programme for Dirichlet Problem for rectangular omain by First Method when $\Phi(p) = x^2 - y^2$.

```
ROGRAMME DIRICHLET (SAXENA,R.S.)
.ARR X(48),Y(48),P(49),Q(49),D(48)X
NP X0(15),Y0(15)X
OM A=0,5 B=1,0 P2=A+B P3=A+2.B P4=2.A+2.B P5=3.A+2.B
6=3.A+3.B P7=3.A+4.B P8=4.A+4.BX
OM :N=12X
.LIB PRO 25(N,AN)X
RI AT HSP :NX
OM H=P8:(2.AN)X
RR C(2304 N.N)X
.COM NP=(2.M-1).HX
F NP (=A THEN 4X
F NP (=P2 THEN 5X
F NP (=P3 THEN 6X
F NP (=P4 THEN 7X
F NP (=P5 THEN 8X
F NP (=P6 THEN 9X
F NP (=P7 THEN 10X
F NP (=P8 THEN 11X
.COM X/K/=B Y/K/=NPX
UMP 12X
.COM X/K/=P2-NP Y/K/=AX
UMP 12X
.COM X/K/=P2-NP Y/K/=AX
UMP 12X
.COM X/K/=-B Y/K/=P4-NPX
UMP 12X
.COM X/K/=-B Y/K/=P4-NPX
UMP 12X
```

```
9.COM X/K/=NP-P6 Y/K/=-AX̄
JUMP 12X̄
10.COM X/K/=NP-P6 Y/K/=-AX̄
JUMP 12X̄
11.COM X/K/=B Y/K/=NP-P8X̄
12.COM XN=X/K/ YN=Y/K/X̄
PRI AT HSP NP,XN,YNX̄
DO 3 M=1 (1).K=1 (1) NX̄
13.COM IP=2.(M-1).HX̄
IF IP (=A THEN 14X̄
IF IP (=P2 THEN 15X̄
IF IP (=P3 THEN 16X̄
IF IP (=P4 THEN 17X̄
IF IP (=P5 THEN 18X̄
IF IP (=P6 THEN 19X̄
IF IP (=P7 THEN 20X̄
IF IP (=P8 THEN 21X̄
14.COM P/K/=B Q/K/=IPX̄
JUMP 22X̄
15.COM P/K/=P2-IP Q/K/=AX̄
JUMP 22X̄
16.COM P/K/=P2-IP Q/K/=AX̄
JUMP 22X̄
17.COM P/K/=-B Q/K/=P4-IPX̄
JUMP 22X̄
18.COM P/K/=-B Q/K/=P4-IPX̄
JUMP 22X̄
19.COM P/K/=IP-P6 Q/K/=-AX̄
JUMP 22X̄
20.COM P/K/=IP-P6 Q/K/=-AX̄
JUMP 22X̄
21.COM P/K/=B Q/K/=IP-P8X̄
22.COM PN=P/K/ QN=Q/K/X̄
PRI AT HSP IP,PN,QNX̄
```

```
DO 13 M=1 (1).K=1 (1) NX̄
COM :NN=N+1X̄
39.COM P/K/=P/1/ Q/K/=Q/1/X̄
DO 39 K=NN (1).1X̄
23.COM D/I/=-(X/I/'2-Y/I/'2)X̄
24.COM FX=LN((P/K/-X/I/)'2+(Q/K/-Y/I/)'2) FY=LN((P/K+1/-X/I/)'2
+(Q/K+1/-Y/I/)'2) XY=(X/K/-X/I/)'2+(Y/K/-Y/I/)'2X̄
IF XY =0 THEN 25 OTH 26X̄
25.COM C/I,K/=2.H.(LN(H)-1)X̄
JUMP 27X̄
26.COM FZ=LN(XY)X̄
COM C/I,K/=H.(FX+FY+4.FZ):6X̄
27.DO 24 K=1 (1) NX̄
DO 23 I=1 (1) NX̄
ALG EQU SYS (N,C,D)X̄
PRI AT HSP D(N)X̄
PRI TEX COORDINATES OF POINT    AV    CV    ERRORX̄
28.COM AV=XO/J/'2-YO/J/'2 SUM=0X̄
29.COM FX=LN((P/K/-XO/J/)'2+(Q/K/-YO/J/)'2) FY=LN((P/K+1/-XO/J/
)'2+(Q/K+1/-YO/J/)'2) FZ=LN((X/K/-XO/J/)'2+(Y/K/-YO/J/)'2)
MD=FX+FY+4.FZ SUM=SUM+MD.D/K/X̄
DO 29 K=1 (1) NX̄
COM CV=-H.SUM:6 ERROR=MOD(AV-CV) PXO=XO/J/ PYO=YO/J/X̄
PRI TAB 7 DIG 13 PXO,13 PYO,12 AV,12 CV,12 ERRORX̄
DO 28 J=1 (1) 15X̄
COM :N=N+12X̄
IF :N )48 THEN 30X̄
JUMP 2X̄
30.STOP X̄
START 1X̄
```

Autocode programme for Dirichlet Problem for rectangular domain by Second Method when g(p) = x.

```
PROGRAMME DIRICHLET (SAXENA,R.S.)X̄
1.ARR X(48),Y(48),P(49),Q(49),D(48)X̄
INP XO(15),YO(15)X̄
COM A=0,5 B=1,0 P2=A+B P3=A+2.B P4=2.A+2.B P5=3.A+2.B
P6=3.A+3.B P7=3.A+4.B P8=4.A+4.B PI=3,141593X̄
COM :N=12X̄
32.LIB PRO 25(N,AN)X̄
PRI AT HSP :NX̄
ARR C(2304 N.N)X̄
COM H=P8:(2.AN)X̄
33.COM LS=(2.M-1).HX̄
PERF 20X̄
COM X/K/=CN Y/K/=DNX̄
DO 33 M=1 (1).K=1 (1) NX̄
34.COM LS=2.(M-1).HX̄
PERF 20X̄
COM P/K/=CN Q/K/=DNX̄
DO 34 M=1 (1).K=1 (1) NX̄
COM :NN=N+1X̄
35.COM P/J/=P/1/ Q/J/=Q/1/X̄
DO 35 J=NN (1).1X̄
42.COM PSM=0X̄
36.COM FX=(Q/J+1/-Y/I/).(P/J/-X/I/)-(Q/J/-Y/I/).(P/J+1/-X/I/)
FY=(Q/J+1/-Y/I/).(Q/J/-Y/I/)+(P/J+1/-X/I/).(P/J/-X/I/)
FI=ARCTG(FX:FY) C/I,J/=FI PSM=PSM+FIX̄
DO 36 J=1 (1) NX̄
PRI AT HSP PSMX̄
COM D/I/=2.PI.X/I/X̄
DO 42 I=1 (1) NX̄
37.COM C/I,J/=C/I,J/+PIX̄
```

```
DO 37 I=1 (1).J=1 (1) NX̄
ALG EQU SYS (N,C,D)X̄
PRI AT HSP D(N)X̄
PRI AT TEL :NX̄
PRI TEX COORDINATES OF POINTS       AV        CV       ERRORX̄
38.COM AV=XO/I/ SUM=0 PSM=0X̄
39.COM FX=(Q/J+1/-YO/I/).(P/J/-XO/I/)-(P/J+1/-XO/I/).(Q/J/-YO/I/
) FY=(Q/J+1/-YO/I/).(Q/J/-YO/I/)+(P/J+1/-XO/I/).(P/J/-XO/I/)X̄
IF FY =0 THEN 66 OTH 88X̄
66.COM SI=PI:2X̄
PRI AT HSP SIX̄
JUMP 99X̄
88.COM SI=ARCTG(FX:FY)X̄
99.IF SI (0 THEN 77 OTH 44X̄
77.COM SI=PI+SIX̄
44.COM SUM=SUM+SI.D/J/ PSM=PSM+SIX̄
DO 39 J=1 (1) NX̄
PRI AT HSP PSMX̄
COM CV=SUM:(2.PI) PXO=XO/I/ PYO=YO/I/ ERROR=MOD(AV-CV)X̄
PRI TAB 7 DIG 13 PXO,13 PYO,12 AV,12 CV,12 ERRORX̄
DO 38 I=1 (1) 15X̄
COM :N=N+12X̄
IF :N) 48 THEN 30 OTH 32X̄
30.STOP X̄
20.SUBROUTINE X̄
IF LS (=A THEN 4X̄
IF LS (=P2 THEN 5X̄
IF LS (=P3 THEN 6X̄
IF LS (=P4 THEN 7X̄
IF LS (=P5 THEN 8X̄
IF LS (=P6 THEN 9X̄
IF LS (=P7 THEN 10X̄
IF LS (=P8 THEN 11X̄
4.COM CN=B DN=LSX̄
```

```
JUMP 12X̲̅
5.COM CN=P2-LS DN=AX̲̅
JUMP 12X̲̅
6.COM CN=P2-LS DN=AX̲̅
JUMP 12X̲̅
7.COM CN=-B DN=P4-LSX̲̅
JUMP 12X̲̅
8.COM CN=-B DN=P4-LSX̲̅
JUMP 12X̲̅
9.COM CN=LS-P6 DN=-AX̲̅
JUMP 12X̲̅
10.COM CN=LS-P6 DN=-AX̲̅
JUMP 12X̲̅
11.COM CN=B DN=LS-P8X̲̅
12.PRI AT HSP LS,CN,DNX̲̅
EXIT X̲̅
START 1X̲̅
```

APPENDIX IV

Autocode programme for torsion problem for equilateral triangular cross-section, by First Method.

```
PROGRAMME TORSION (SAXENA,R.S.)X̄
1.ARR X(48),Y(48),P(49),Q(49),D(48)X̄
INP XO(56),YO(56)X̄
PRI TEX TORSION PROBLEM FOR TRIANGLE BY FIRST METHODX̄
COM :M=2X̄
2.LIB PRO 25(M,AM)X̄
COM :M2=2.M N=6.M MM=M+1 MK=3.M MD=MK+1 NN=N+1 MN=MM+1
MS=M2+1 MP=MD+1X̄
LIB PRO 25(M2,AM2)X̄
COM A=1:3'(1:2) H=0,5:AM H1=H.3'(1:2):2 PI=3.141593X̄
ARR C(2304 N.N)X̄
PRI AT HSP :NX̄
3.COM X/I/=A Y/I/=(2.AK-1).HX̄
DO 3 AK=1 (1).I=1 (1) MX̄
4.COM X/I/=A-(2.AI-1).H1 Y/I/=(X/I/+2.A):3'(1:2)X̄
DO 4 AI=1 (1).I=MM (1) MKX̄
5.COM X/I/=X/J/ Y/I/=-Y/J/X̄
DO 5 I=MD (1).J=MK (1).1X̄
COM :MK=MK-1 MD=MD+1X̄
IF :1-MK (=0 THEN 5X̄
PRI AT HSP :N,X(N),Y(N)X̄
6.COM P/I/=A Q/I/=2.AK.HX̄
DO 6 AK=0 (1).I=1 (1) MMX̄
7.COM P/I/=A-2.AI.H1 Q/I/=(P/I/+2.A):3'(1:2)X̄
DO 7 AI=1 (1).I=MN (1) MDX̄
COM :MK=3.MX̄
8.COM P/I/=P/J/ Q/I/=-Q/J/X̄
DO 8 I=MP (1).J=MK (1).1X̄
COM :MK=MK-1 MP=MP+1X̄
```

```
IF :1-MK (=0 THEN 8X̄
PRI AT HSP P(NN),Q(NN)X̄
23.COM D/I/=-(X/I/'2+Y/I/'2):2X̄
24.COM FX=LN((P/K/-X/I/)'2+(Q/K/-Y/I/)'2) FY=LN((P/K+1/-X/I/)'2+
(Q/K+1/-Y/I/)'2) XY=(X/K/-X/I/)'2+(Y/K/-Y/I/)'2X̄
IF XY =0 THEN 25 OTH 26X̄
25.COM C/I,K/=2.H.(LN(H)-1)X̄
JUMP 27X̄
26.COM FZ=LN(XY)X̄
COM C/I,K/=H.(FX+FY+4.FZ):6X̄
27.DO 24 K=1 (1) NX̄
DO 23 I=1 (1) NX̄
ALG EQU SYS (N,C,D)X̄
PRI AT HSP D(N)X̄
PRI AT TEL :NX̄
PRI TEX COORDINATES OF POINTS    AV    CV     ERRORX̄
28.COM AV=-(XO/J/'3-3.XO/J/.YO/J/'2):(6.A)+2.A'2:3 SUM=0X̄
29.COM FX=LN((P/K/-XO/J/)'2+(Q/K/-YO/J/)'2) FY=LN((P/K+1/-XO/J/
)'2+(Q/K+1/-YO/J/)'2) FZ=LN((X/K/-XO/J/)'2+(Y/K/-YO/J/)'2)
MD=FX+FY+4.FZ SUM=SUM+MD.D/K/X̄
DO 29 K=1 (1) NX̄
COM CV=-H.SUM:6 ERROR=MOD(AV-CV) PXO=XO/J/ PYO=YO/J/X̄
COM CONST=CV-(PXO'2+PYO'2):2X̄
PRI TAB 7 DIG 10 PXO,10 PYO,12 AV,12 CV,12 ERROR,10 CONSTX̄
DO 28 J=1 (1) 56X̄
COM :M=M+2X̄
IF :M )8 THEN 20 OTH 2X̄
20. STOP X̄
START 1X̄
```

Autocode programme for torsion problem for equilateral triangular cross-section, by Second Method

```
PROGRAMME TORSION (SAXENA,R.S.)X̄
```

```
1.ARR X(48),Y(48),P(49),Q(49),D(48)X̄
INP XO(56),YO(56)X̄
COM :M=2X̄
2.LIB PRO 25(M,AM)X̄
COM :M2=2.M N=6.M MM=M+1 MK=3.M MD=MK+1 NN=N+1 MN=MM+1
MS=M2+1 MP=MD+1X̄
COM A=1:3'(1:2) H=0,5:AM HI=H.3'(1:2):2 PI=3.141593X̄
ARR C(2304 N.N)X̄
PRI AT HSP :NX̄
3.COM X/I/=A Y/I/=(2.AK-1).HX̄
DO 3 AK=1 (1).I=1 (1) MX̄
4.COM X/I/=A-(2.AI-1).HI Y/I/=(X/I/+2.A):3'(1:2)X̄
DO 4 AI=1 (1).I=MM (1) MKX̄
5.COM X/I/=X/J/ Y/I/=-Y/J/X̄
DO 5 I=MD (1).J=MK (1).1X̄
COM :MK=MK-1 MD=MD+1X̄
IF :1-MK (=0 THEN 5X̄
PRI AT HSP X(N),Y(N)X̄
6.COM P/I/=A Q/I/=2.AK.HX̄
DO 6 AK=0 (1).I=1 (1) MMX̄
7.COM P/I/=A-2.AI.HI Q/I/=(P/I/+2.A):3'(1:2)X̄
DO 7 AI=1 (1).I=MN (1) MDX̄
COM :MK=3.MX̄
8.COM P/I/=P/J/ Q/I/=-Q/J/X̄
DO 8 I=MP (1).J=MK (1).1X̄
COM :MK=MK-1 MP=MP+1X̄
IF :1-MK (=0 THEN 8X̄
PRI AT HSP P(NN),Q(NN)X̄
19.COM SUM=0X̄
9.COM FX=-(P/J+1/-X/I/).(Q/J/-Y/I/)+(Q/J+1/-Y/I/).(P/J/-X/I/)
FY=(P/J+1/-X/I/).(P/J/-X/I/)+(Q/J+1/-Y/I/).(Q/J/-Y/I/)X̄
COM FYM=MOD(FY)X̄
IF FYM (-8 THEN 66 OTH 88X̄
66.COM SI=PI:2X̄
```

```
JUMP 44 X̄
88.COM SI=ARCTG(FX: FY)X̄
44.COM C/I,J/=SI SUM=SUM+SIX̄
DO 9 J=1 (1) NX̄
PRI AT HSP SUMX̄
COMP D/I/=PI.(X/I/'2+Y/I/'2)X̄
DO 19 I=1 (1) NX̄
10.COMP C/I,J/=C/I,J/+PIX̄
DO 10 I=1 (1) .J=1 (1) NX̄
ALG EQU SYS (N,C,D)X̄
PRI AT HSP D(N)X̄
PRI AT TEL :NX̄
PRI TEXT COORDINATES OF POINTS     AV     CV     ERRORX̄
11.COM AV=-(XO/I/'3-3.XO/I/.YO/I/'2):(6.A)+2.A'2:3 SUM=OX̄
COM PSM=OX̄
12.COMP FX=-(P/J+1/-XO/I/).(Q/J/-YO/I/)+(Q/J+1/-YO/I/).
(P/J/-XO/I/) FY=(P/J+1/-XO/I/).(P/J/-XO/I/)+(Q/J+1/-YO/I/).
(Q/J/-YO/I/)X̄
IF FY=0 THEN 60 OTHERWISE 80X̄
60.COM SI=PI:2X̄
JUMP 90X̄
80.COM SI=ARCTG(FX:FY)X̄
90.IF SI (O THEN 70 OTHERWISE 40X̄
70.COM SI=PI+SIX̄
40.COM SUM=SUM+SI.D/J/ PSM=PSM+SIX̄
DO 12 J=1 (1) NX̄
PRI AT HSP PSMX̄
COMP CV=SUM:(2.PI) ERROR=MOD(AV-CV) PXO=XO/I/ PYO=YO/I/X̄
COM CONST=CV-(PXO 2+PYO'2):2X̄
PRI TAB 7 DIG 10 PXO,10 PYO,12 AV,12 CV,12 ERROR,10 CONSTX̄
DO 11 I=1 (1) 56X̄
COMP :M=M+2X̄
IF :M )8 THEN 20 OTHERWISE 2X̄
20. STOP X̄
START 1X̄
```

APPENDIX V

Autocode programme for torsion problem for circular cross-section with a circular notch in the ratio a:b = 4:1, by First Method.

```
PROGRAMME TORSION (SAXENA,R.S.)X̄
1.ARR X(36),Y(36),P(37),Q(37),D(36)X̄
INP XO(101),YO(101)X̄
COM A=2,0 B=0,5 THETA=ARCCOS(B:(2.A)) PI=3,141593X̄
COM :M=1X̄
2.LIB PRO 25(M,AM)X̄
COM :N=8.M NN=N+1 MM=M+1 MN=M+N KM=MN+1 TM=2.MN DM=MN+2
IN=N+2.M MV=TM+1 CN=IN+1X̄
ARR C(1296 TM.TM)X̄
COM FI=THETA:AM SM=AM+1 H=B.FI:2X̄
3.COM X/I/=A.(1+COS((2.AK-1).FI:8)) Y/I/=A.SIN((2.AK-1).FI:8)
P/I/=A.(1+COS((AK-1).FI:4)) Q/I/=A.SIN((AK-1).FI:4)X̄
DO 3 AK=1 (1).I=1 (1) NX̄
4.COM X/I/=B.COS((2.AJ-1).FI:2) Y/I/=B.SIN((2.AJ-1).FI:2)X̄
DO 4 AJ=AM (-1).I=NN (1).MX̄
5.COM X/K/=X/J/ Y/K/=-Y/J/X̄
DO 5 J=MN (1).K=KM (1).1X̄
COM :MN=MN-1 KM=KM+1X̄
IF :KM-TM (=0 THEN 5X̄
PRI AT HSP :TM,X(TM),Y(TM)X̄
6.COM P/I/=B.COS((AJ-1).FI) Q/I/=B.SIN((AJ-1).FI)X̄
DO 6 AJ=SM (-1).I=NN (1).MMX̄
COM :MN=M+NX̄
7.COM P/I/=P/J/ Q/I/=-Q/J/X̄
DO 7 J=MN (1).I=DM (1).1X̄
COM :MN=MN-1 DM=DM+1X̄
IF :DM-MV (=0 THEN 7X̄
PRI AT HSP P(MV),Q(MV)X̄
```

```
COM :NI=0X̄
9.COM :NI=NI+1 NK=0X̄
10.COM :NK=NK+1X̄
PERF 20X̄
IF FZ =0 THEN 15 OTH 25X̄
15.COM R=AX̄
PERF 30X̄
COM C/I,K/=INVX̄
JUMP 40X̄
25.COM ZN=LN(FZ):2X̄
COM C/I,K/=(XN+YN+4.ZN).H:3X̄
40.DO 10 K=1 (1) TMX̄
DO 9 I=1 (1) NX̄
COM :NI=NN-1X̄
49.COM :NI=NI+1 NK=0X̄
50.COM :NK=NK+1X̄
PERF 20X̄
IF FZ =0 THEN 55 OTH 45X̄
55.COM R=BX̄
PERF 30X̄
COM C/I,K/=INVX̄
JUMP 60X̄
45.COM ZN=LN(FZ):2X̄
COM C/I,K/=(XN+YN+4.ZN).H:3X̄
60.DO 50 K=1 (1) TMX̄
DO 49 I=NN (1) INX̄
COM :NI=CN-1X̄
69.COM :NI=NI+1 NK=0X̄
70.COM :NK=NK+1X̄
PERF 20X̄
IF FZ =0 THEN 75 OTH 80X̄
75.COM R=AX̄
PERF 30X̄
COM C/I,K/=INVX̄
```

```
JUMP 90X̲
80.COM ZN=LN(FZ):2X̲
COM C/I,N/=(XN+YN+4.ZN).H:3X̲
90.DO 70 K=1 (1) TMX̲
DO 69 I=C.N (1) TMX̲
85.COM D/I/=-(X/I/'2+Y/I/'2):2X̲
DO 85 I=1 (1) TMX̲
ALG EQU SYS (TM,C,D)X̲
PRI AT HSP D(TM)X̲
PRI AT TEL :M,:NX̲
PRI TEXT COORDINATES OF POINTS        AV        CV        ERRORX̲
95.COM AV=A.(X0/I/-B'2.X0/I/:(X0/I/'2+Y0/I/'2))+B'2:2 SUM=0X̲
96.COM FX=LN((P/J+1/-X0/I/)'2+(Q/J+1/-Y0/I/)'2) FY=LN((P/J/-
X0/I/)'2+(Q/J/-Y0/I/)'2) FZ=LN((X/J/-X0/I/)'2+(Y/J/-Y0/I/)'2)
MF=FX+FY+4.FZ SUM=SUM+MF.D/J/X̲
DO 96 J=1 (1) TMX̲
COM CV=-H.SUM:6 ERROR=MOD(AV-CV) PX0=X0/I/ PY0=Y0/I/X̲
COM CONST=CV-(PX0'2+PY0'2):2X̲
PRI TAB 7 DIG 10 PX0,10 PY0,12 AV,12 CV,12 ERROR,10 CONSTX̲
DO 95 I=1 (1) 103X̲
COM :M=2.MX̲
IF :M)2 THEN 99 OTH 2X̲
99.STOP X̲
20.SUBROUTINE X̲
100.COM XN=LN((P/L+1/-X/J/)'2+(Q/L+1/-Y/J/)'2):2
YN=LN((P/L/-X/J/)'2+(Q/L/-Y/J/)'2):2
FZ=(X/L/-X/J/)'2+(Y/L/-Y/J/)'2X̲
DO 100 L=NK (1).J=NI (1).1X̲
EXIT X̲
30.SUBROUTINE X̲
COM APPV=(LN(H)-1)-H'2:(72.R'2)-H'4:(14400.R'4)-
H'6:(1270080.R'6) INT=H.APPV INV=2.INTX̲
EXIT X̲
START 1X̲
```

Autocode programme for torsion problem for curcular cross-section with a circular notch in the ratio a:b = 8:1, by Second Method.

```
PROGRAMME TORSION (SAXENA,R.S.)X̄
1.ARR X(34),Y(34),P(35),Q(35),D(34)X̄
COM A=2,0 B=0,25 THETA=ARCCOS(B:(2.A)) PI=3,141593X̄
INP XO(103),YO(103)X̄
PRI TEX TOR.PRO FOR CIRCLE WITH A NOTCH BY SECOND METHODX̄
COM :M=1X̄
2.LIB PRO 25(M,AM)X̄
COM :N=16.M NN=N+1 MM=M+1 MN=M+N KM=MN+1 TM=2.MN
DM=MN+2 IN=N+2.M MV=TM+1 CN=IN+1X̄
ARR C(1156 TM.TM)X̄
COM FI=THETA:AM SM=AM+1 H=B.FI:2X̄
3.COM X/I/=A.(1+COS((2.AK-1).FI:16) Y/I/=A.SIN((2.AK-1).FI:16
P/I/=A.(1+COS((AK-1).FI:8)) Q/I/=A.SIN((AK-1).FI:8)X̄
DO 3 AK=1 (1).I=1 (1) NX̄
4.COM X/I/=B.COS((2.AJ-1).FI:2) Y/I/=B.SIN((2.AJ-1).FI:2)X̄
DO 4 AJ=AM (-1).I=MN (1).MX̄
5.COM X/K/=X/J/ Y/K/=-Y/J/X̄
DO 5 J=MN (1).K=KM (1).1X̄
COM :MN=MN-1 KM=KM+1X̄
IF :KM-TM (=0 THEN 5X̄
PRI AT HSP :TM,X(TM),Y(TM)X̄
6.COM P/I/=B.COS((AJ-1).FI) Q/I/=B.SIN((AJ-1).FI)X̄
DO 6 AJ=SM (-1).I=NN (1).MMX̄
COM :MN=M+NX̄
7.COM P/I/=P/J/ Q/I/=-Q/J/X̄
DO 7 J=MN (1).I=DM (1).1X̄
COM :MN=MN-1 DM=DM+1X̄
IF :DM-MV (=0 THEN 7X̄
PRI AT HSP P(MV),Q(MV)X̄
COM :KN=0X̄
```

```
8.COM :KN=KN+1 JN=0X̄
COM SUM=0X̄
10.COM :JN=JN+1X̄
IF :JN-KN =0 THEN 25 OTH 35X̄
25.PERF 30X̄
COM C/I,J/=SIX̄
JUMP 40X̄
35.PERF 20X̄
COM C/I,J/=SIX̄
40.COM SUM=SUM+SIX̄
DO 10 J=1 (1) TMX̄
PRI AT HSP SUMX̄
DO 8 I=1 (1) NX̄
COM :NN=NX̄
15.COM :KN=KN+1 JN=0X̄
COM SUM=0X̄
45.COM :JN=JN+1X̄
IF :JN-KN =0 THEN 55 OTH 65X̄
55.PERF 100X̄
COM C/I,J/=SIX̄
JUMP 50X̄
65.PERF 20X̄
COM C/I,J/=SIX̄
50.COM SUM=SUM+SIX̄
DO 45 J=1 (1) TMX̄
PRI AT HSP SUMX̄
DO 15 I=NN (1) INX̄
COM :KN=INX̄
70.COM :KN=KN+1 JN=0X̄
COM SUM=0X̄
75.COM :JN=JN+1X̄
IF :JN-KN =0 THEN 85 OTH 95X̄
85.PERF 30X̄
COM C/I,J/=SIX̄
```

```
JUMP 80X̄
95.PERF 80X̄
COM C/I,J/=SIX̄
80.COM SUM=SUM+SIX̄
DO 75 J=1 (1) TMX̄
PRI AT HSP SUMX̄
DO 70 I=CN (1) TMX̄
66.COM C/I,J/=C/I,J/+PI D/I/=PI.(X/I/'2+Y/I/'2)X̄
DO 66 J=1 (1).I=1 (1) TMX̄
ALG EQU SYS (TM,C,D)X̄
PRI AT HSP D(TM)X̄
PRI AT TEL :M,:NX̄
PRI TEXT COORDINATES OF POINTS    AV    CV    ERROR    CONSTX̄
68.COM AV=A.(X0/I/-B'2.X0/I/:(X0/I/'2+Y0/I/'2))+B'2:2 SUM=0X̄
COM PSM=0X̄
69.COM FX=(Q/J+1/-Y0/I/).(P/J/-X0/I/)-(P/J+1/-X0/I/).(Q/J/-
Y0/I/) FY=(P/J+1/-X0/I/).(P/J/-X0/I/)+(Q/J/-Y0/I/).(Q/J+1/-
Y0/I/)X̄
COM FYM=MOD(FY)X̄
IF FYM (10-8 THEN 104 OTH 105X̄
104.COM SI=PI:2X̄
JUMP 92X̄
105.IF FY (0 THEN 107 OTH 102X̄
107.COM XY=ARCTG(FX:FY) SI=PI+XYX̄
JUMP 92X̄
102.COM SI=ARCTG(FX:FY)X̄
92.COM SUM=SUM+SI.D/J/ PSM=PSM+SIX̄
DO 69 J=1 (1) TMX̄
PRI AT HSP PSMX̄
COM CV=SUM:(2.PI) ERROR=MOD(AV-CV) PX0=X0/I/ PY0=Y0/I/X̄
COM CONST=CV-(PX0'2+PY0'2):2X̄
PRI TABLE 7 DIG 10 PX0,10 PY0,12 AV,12 CV,12 ERROR,10 CONSTX̄
DO 68 I=1 (1) 94X̄
99.STOP X̄
```

```
20.SUBROUTINE X̄
43.COM FX=(Q/L+1/-Y/K/).(P/L/-X/K/)-(P/L+1/-X/K/).(Q/L/-Y/K/)
FY=(P/L+1/-X/K/).(P/L/-X/K/)+(Q/L/-Y/K/).(Q/L+1/-Y/K/)
SI=ARCTG(FX:FY)X̄
DO 43 K=KN (1).L=JN (1).1X̄
EXIT X̄
30.SUBROUTINE X̄
53.COM FX=A.(P/K/+X/K/)-(P/K/.X/K/+Q/K/.Y/K/) FY=A.(Q/K/-Y/K/)+
(Y/K/.P/K/-X/K/.Q/K/) SI=2.ARCTG(FX:FY)X̄
DO 53 K=KN (1).1X̄
EXIT X̄
100.SUBROUTINE X̄
101.COM FX=(X/K/.P/K/+Y/K/.Q/K/)-B'2 FY=X/K/.Q/K/-Y/K/.P/K/
SI=2.ARCTG(FX:FY)X̄
DO 101 K=KN (1).1X̄
EXIT X̄
START 1X̄
```

APPENDIX VI

Autocode programme for computation of stress function for rectangular cross-section (2 X 1) with a rectangular notch(.4 X .2) by First Method.

```
PROGRAMME TORSION (SAXENA,R.S.)X̄
1.ARR X(48),Y(48),P(49),Q(49),D(48),CV(84),XO(84),YO(84)X̄
COM :XN=1X̄
COM QP=0,1X̄
21.COM XO/I/=0,1.AI YO/I/=QPX̄
DO 21 AI=0 (1).I=XN (1).10X̄
COM QP=QP+0,1X̄
COM :XN=XN+10X̄
IF :XN (70 THEN 21X̄
22.COM XO/I/=0,1.AI YO/I/=QPX̄
DO 22 AI=3 (1).I=XN (1).7X̄
COM QP=QP+0,1X̄
COM :XN=XN+7X̄
IF :XN (84 THEN 22X̄
PRI AT HSP XO(84),YO(84)X̄
COM :N=0X̄
2.COM :N=N+16X̄
LIB PRO 25(N,AN)X̄
ARR C(2304 N.N)X̄
COM HP=3,2:AN H=2.HP PI=3,141593 BN=AN:2X̄
LIB PRO 26(BN,NK)X̄
COM :MK=NK+1 PK=NKX̄
3.COM LS=(2.M-1).HPX̄
PERF 60X̄
COM X/K/=CN Y/K/=DNX̄
DO 3 M=1 (1).K=1 (1) NKX̄
5.COM X/I/=-X/J/ Y/I/=Y/J/X̄
DO 5 I=MK (1).J=NK (1).1X̄
```

```
COM :MK=MK+1 NK=NK-1X̄
IF :MK-N (=0 THEN 5X̄
PRI AT HSP :N,X(N),Y(N)X̄
4.COM LS=2.(M-1).HPX̄
PERF 60X̄
COM P/K/=CN Q/K/=DNX̄
DO 4 M=1 (1).K=1 (1) MKX̄
COM :NK=PK CK=NK+2 MN=N+1X̄
9.COM P/I/=-P/J/ Q/I/=Q/JX̄
DO 9 I=CK (1).J=NK (1).1X̄
COM :CK=CK+1 NK=NK-1X̄
IF :CK-MN (=0 THEN 9X̄
PRI AT HSP P(NN),Q(NN)X̄
10.COM D/I/=-(X/I/'2+Y/I/'2):2X̄
11.COM FX=LN((P/K/-X/I/)'2+(Q/K/-Y/I/)'2) FY=LN((P/K+1/-
X/I/)'2+(Q/K+1/-Y/I/)'2) XY=(X/K/-X/I/)'2+(Y/K/-Y/I/)'2X̄
IF XY =0 THEN 12 OTH 13X̄
12.COM C/I,K/=H.(LN(H:2)-1)X̄
JUMP 16X̄
13.COM FZ=LN(XY) C/I,K/=H.(FX+FY+4.FZ):12X̄
16.DO 11 K=1 (1) NX̄
DO 10 I=1 (1) NX̄
ALG EQU SYS (N,C,D)X̄
PRI AT HSP D(N)X̄
PRI TEX TOR.PRO. FOR RECTANGLE WITH  A RECTANGULAR NOTCHX̄
PRI AT TEL :NX̄
PRI TEXT COORDINATES OF POINTS    VALUES OF SI    STRESS FUNCTIONX̄
14.COM SUM=0 PSM=0X̄
15.COM FX=LN((P/K/-X0/I/)'2+(Q/K/-Y0/I/)'2) FY=LN((P/K+1/-
X0/I/)'2+(Q/K+1/-Y0/I/)'2) FZ=LN((X/K/-X0/I/)'2+(Y/K/-Y0/I/)'2)
MD=FX+FY+4.FZ SUM=SUM+MD.D/K/X̄
DO 15 K=1 (1) NX̄
COM CV/I/=-H.SUM:12 CVX=CV/I/ PX0=X0/I/ PY0=Y0/I/
CONST=CVX-(PX0'2+PY0'2):2X̄
```

```
PRI TAB 7 DIG 13 PXO,13 PYO,15 CVX,15 CONSTX
DO 14 I=1 (1) 84X
IF :N )48 THEN 70 OTH 2X
70.STOP X
60.SUB X
IF LS (=1 THEN 61X
IF LS (=2 THEN 62X
IF LS (=2,8 THEN 63X
IF LS (=3,0 THEN 64 OTH 65X
61.COM CN=LS DN=0X
JUMP 66X
62.COM CN=1 DN=LS-1X
JUMP 66X
63.COM CN=3-LS DN=1X
JUMP 66X
64.COM CN=0,2 DN=3,8-LSX
JUMP 66X
65.COM CN=3,2-LS DN=0,8X
66.PRI AT HSP LS,CN,DNX
EXIT X
START 1X
```

Autocode programme for computation of stress function for rectangular cross-section (2 X 1) with a equilateral triangular notch by Second Method.

```
PROGRAMME TORSION (SAXENA,R.S.)X
1.ARR X(48),Y(48),P(49),Q(49),D(48),CV(86),XO(86),YO(86)X
COM QP=0,1X
COM :XN=1X
21.COM XO/I/=0,1.AI YO/I/=QPX
DO 21 AI=0 (1).I=XN (1).10X
COM QP=QP+0,1X
COM :XN=XN+10X
```

```
IF :XN (52 THEN 21X
COM QP=0,7X
22.COM XO/I/=0,1.AI YO/I/=QPX
DO 22 AI=1 (1).I=XN (1).9X
COM QP=QP+0,1X
COM :XN=XN+9X
IF :XN (72 THEN 22X
35.COM XO/I/=0,1.AI YO/I/=0,9X
DO 35 AI=2 (1).I=XN (1) 86X
COM :N=0X
2.COM :N=N+16X
LIB PRO 25(N,AN)X
ARR C(2304 N.N)X
COM HP=3,2:AN H=2.HP PI=3,141593 BN=AN:2X
LIB PRO 26(BN,NK)X
COM :MK=NK+1 PK=NKX
3.COM LS=(2.M-1).HPX
PERF 60X
COM X/K/=CN Y/K/=DNX
DO 3 M=1 (1).K=1 (1) NKX
5.COM X/I/=-X/J/ Y/I/=Y/J/X
DO 5 I=MK (1).J=NK (1).1X
COM :MK=MK+1 NK=NK-1X
IF :MK-N (=0 THEN 5X
PRI AT HSP :N,X(N),Y(N)X
COM :MK=PK+1X
4.COM LS=2.(M-1).HPX
PERF 60X
COM P/K/=CN Q/K/=DNX
DO 4 M=1 (1).K=1 (1) MKX
COM :NK=PK CK=NK+2 NN=N+1X
9.COM P/I/=-P/J/ Q/I/=Q/J/X
DO 9 I=CK (1).J=NK (1).1X
COM :CK=CK+1 NK=NK-1X
```

```
IF :CA-NN (=0 THEN 9X̄
PRI AT HSP P(NN),Q(NN)X̄
10.COM SUM=0X̄
11.COM FX=(Q/J+1/-Y/I/).(P/J/-X/I/)-(Q/J/-Y/I/).(P/J+1/-X/I/)
FY=(Q/J+1/-Y/I/).(Q/J/-Y/I/)+(P/J+1/-X/I/).(P/J/-X/I/)X̄
COM FYM=MOD(FY)X̄
IF FYM (10-8 THEN 67 OTH 88X̄
67.IF Y/I/ =0 THEN 98X̄
COM SI=-PI:2X̄
JUMP 44X̄
98.COM SI=PI:2X̄
JUMP 44 X̄
88.COM SI=ARCTG(FX:FY)X̄
44.COM C/I,J/=SI SUM=SUM+SIX̄
DO 11 J=1 (1) NX̄
PRI AT HSP SUMX̄X
COM D/I/=PI.(X/I/'2+Y/I/'2)X̄
DO 10 I=1 (1) NX̄
12.COM C/I,J/=C/I,J/+PIX̄
DO 12 J=1 (1).I=1 (1) NX̄
ALG EQU SYS (N,C,D)X̄
PRI AT HSP D(N)X̄
PRI TEX TOR.PRO FOR RECTANGLE WITH A TRIANGULAR NOTCHX̄
PRI AT TEL :NX̄
PRI TEXT COORDINATES OF POINTS      SI     CONSTX̄
14.COM SUM=0 PSM=0X̄
15.COM FX=(Q/J+1/-YO/I/).(P/J/-XO/I/)-(Q/J/-YO/I/).(P/J+1/-XO/I/)
FY=(Q/J+1/-YO/I/).(Q/J/-YO/I/)+(P/J+1/-XO/I/).(P/J/-XO/I/)X̄
COM FYM=MOD(FY)X̄
IF FYM (10-8 THEN 59 OTH 69X̄
59.COM FI=PI:2X̄
JUMP 79X̄
69.IF FY (0 THEN 77 OTH 78X̄
77.COM XY=ARCTG(FX:FY) FI=PI+XYX̄
```

```
JUMP 79X̄
78.COM FI=ARCTG(FX:FY)X̄
79.COM SUM=SUM+FI.D/J/ PSM=PSM+FIX̄
DO 15 J=1 (1) NX̄
PRI AT HSP PSMX̄
COM CV/I/=SUM:(2,PI) PXO=XO/I/ PYO=YO/I/X̄
COM CONST=CV/I/-(PXO'2+PYO'2):2 CVX=CV/I/X̄
PRI TAB 7 DIG 13 PXO,13 PYO,15 CVX,15 CONSTX̄
DO 14 I=1 (1) 86X̄
IF :N )32 THEN 70 OTH 2X̄
70.STOP X̄
60.SUB X̄
IF LS (=1 THEN 61X̄
IF LS (=2 THEN 62X̄
IF LS (=2,8 THEN 63 OTH 64X̄
61.COM CN=LS DN=0X̄
JUMP 65X̄
62.COM CN=1 DN=LS-1X̄
JUMP 65X̄
63.COM CN=3-LS DN=1X̄
JUMP 65X̄
64.COM CN=1,6-LS:2 DN=1-(LS-2,8).3'(1:2):2X̄
65.PRI AT HSP LS,CN,DNX̄
EXIT X̄
START 1X̄
```

Autocode programme for computation maximum shearing stress for rectangular cross-section (2 X 1) with a rectangular notch (.4 X .2) .

```
PROGRAMME STRESS (SAXENA,R.S.)X̄
1.ARR SI(12111.11)X̄
INP CV(84)X̄
COM QP=0,1 H=0,2X̄
```

```
23.COM SI/J,1/=((AI-1).QP)'2:2X
DO 23 AI=1 (1).J=1 (1) 11X
24.COM SI/J,11/=(1+((AI-1).QP)'2):2X
DO 24 AI=3 (1).J=3 (1) 11X
25.COM SI/11,J/=(1+((AI-1).QP)'2):2X
DO 25 AI=2 (1).J=2 (1) 10X
26.COM SI/J,9/=(0,64+((AI-1).QP)'2):2X
DO 26 AI=1 (1).J=1 (1) 3X
COM SI/3,10/=0,425X
COM :TI=1X
27.COM SI/J,K/=CV/I/X
DO 27 I=TI (1).J=1 (1) 10X
COM :TI=TI+10X
DO 27 K=2 (1).7X
COM :TI=71X
28.COM SI/J,K/=CV/I/X
DO 28 I=TI (1).J=4 (1) 10X
COM :TI=TI+7X
DO 28 K=9 (1).2X
PRI TEX COOR. OF POINTS   DELX    DELY    TAUZX    TAUYZX
COM YI=0X
29.COM YI=YI+QP XJ=0X
30.COM XJ=XJ+QP DELX=(SI/I+1,J/-SI/I-1,J/):H
DELY=(SI/I,J+1/-SI/I,J-1/):H TAUYZ=XJ-DELX
TAUZX=DELY-YI TAU=(TAUYZ'2+TAUZX'2)'(1:2)
CONST=SI/I,J/-(YI'2+XJ'2):2X
PRI TAB 7 DIG 10 XJ,10 YI,10 DELX,10 DELY,10 TAUZX,10 TAUYZX
PRI AT HSP TAU,CONSTX
DO 30 I=2 (1) 10X
DO 29 J=2 (1) 8X
COM YI=0,7X
31.COM YI=YI+QP XJ=HX
32.COM XJ=XJ+QP DELX=(SI/I+1,J/-SI/I-1,J/):H
DELY=(SI/I,J+1/-SI/I,J-1/):H TAUYZ=XJ-DELX
```

```
TAUZX=DELY-YI TAU=(TAUYZ'2+TAUZX'2)'(1:2)
CONST=SI/I,J/-(YI'2+XJ'2):2X̲
PRI TAB 7 DIG 10 XJ,10 YI,10 DELX,10 DELY,10 TAUZX,10 TAUYZX̲
PRI AT HSP TAU,CONSTX̲
DO 32 I=4 (1) 10X̲
DO 31 J=9 (1) 10X̲
STOP X̲
START 1X̲
```

Autocode programme for computation of maximum shearing stress for rectangular cross-section (2 X 1) with a equilateral triangular notch (.4).

```
PROGRAMME STRESS (SAXENA,R.S.)X̲
1.ARR SI(12111.11)X̲
INP CV(86)X̲
COM QP=0,1 H=0,2X̲
23.COM SI/J,1/=((AI-1).QP)'2:2X̲
DO 23 AI=1 (1).J=1 (1) 11X̲
24.COM SI/11,J/=(1+((AI-1).QP)'2):2X̲
DO 24 AI=2 (1).J=2 (1) 11X̲
25.COM SI/J,11/=(1+((AI-1).QP)'2):2X̲
DO 25 AI=3 (1).J=3 (1) 10X̲
COM :TI=1X̲
27.COM SI/K,J/=CV/I/X̲
DO 27 I=TI (1).K=1 (1) 10X̲
COM :TI=TI+10X̲
DO 27 J=2 (1).6X̲
COM :TI=61X̲
28.COM SI/K,J/=CV/I/X̲
DO 28 I=TI (1).K=2 (1) 10X̲
COM :TI=TI+9X̲
DO 28 J=8 (1) 9X̲
```

```
38.COM SI/K,10/=CV/I/X
DO 38 K=3 (1) .I=79 (1) 86X
PRI TEX COOR. OF POINTS     DELX     DELY     TAUZX     TAUYZX
COM YI=0X
COM :G=1X
29.COM YI=YI+QP XJ=0X
COM :G=G+1 Z=1X
30.COM XJ=XJ+QPX
COM :Z=Z+1X
PERF 40X
COM CONST=SI/J,I/-(YI'2+XJ'2) :2X
PRI AT HSP CONSTX
DO 30 J=2 (1) 10X
DO 29 I=2 (1) 7X
31.COM YI=YI+QP XJ=QPX
COM :G=G+1 Z=2X
32.COM XJ=XJ+QPX
COM :Z=Z+1X
PERF 40X
COM CONST=SI/J,I/-(YI'2+XJ'2) :2X
PRI AT HSP CONSTX
DO 32 J=3 (1) 10X
DO 31 I=8 (1) 9X
33.COM YI=YI+QP XJ=HX
COM :G=G+1 Z=3X
34.COM XJ=XJ+QPX
COM :Z=Z+1X
PERF 40X
COM CONST=SI/J,I/-(YI'2+XJ'2) :2X
PRI AT HSP CONSTX
DO 34 J=4 (1) 10X
DO 33 I=10 (1) .1X
STOP X
```

```
40.SUBROUTINE X̄
41.COM FX=(SI/K+1,L/-SI/K-1,L/):H FY=
(SI/K,L+1/-SI/K,L-1/):H TAUYZ=XJ-FX
TAUZX=FY-YI TAU=(TAUYZ'2+TAUZX'2)'(1:2)X̄
PRI AT ABP TAUX̄
PRI TAB 7 DIG 8 XJ,8 YI,10 FX,10 FY,10 TAUZX,10 TAUYZX̄
DO 41 K=Z (1).L=G (1).1X̄
EXIT X̄
START 1X̄
```

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